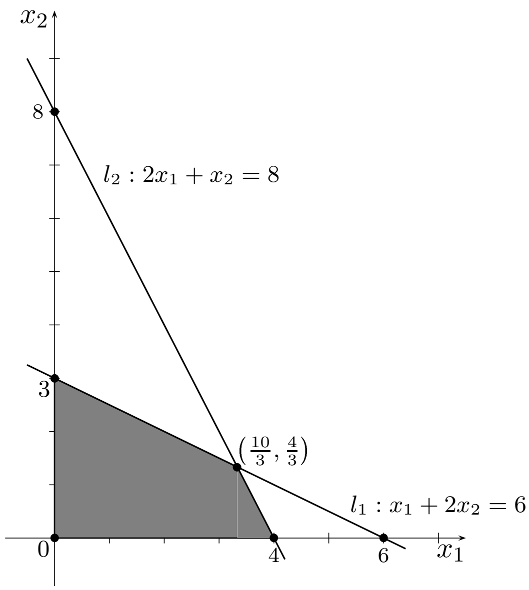
Problem: Maximize $z = 3x_1 + 4x_2$ subject to $\begin{cases} x_1 + 2x_2 & \le 6 \\ 2x_1 + x_2 & \le 8 \\ x_1\ ,\ x_2 & \ge 0 \end{cases}$

Graphical Solution: (can be applied when graphs exist - this is not always the case so the technique is limited)



$\left(\frac{10}{3}, \frac{4}{3}\right)$ is the intersection of l_1 and l_2
The shaded region corresponds to the given inequalities

Corners of Region	$z = 3x_1 + 4x_2$
$(0,0)$	$0 \longleftarrow$ minimum value of z
$(0,3)$	12
$(4,0)$	12
$\left(\frac{10}{3}, \frac{4}{3}\right)$	$\frac{46}{3} = 15\frac{1}{3} \longleftarrow$ maximum value of z

$$z_{max} = \frac{46}{3} \text{ at } \left(\frac{10}{3}, \frac{4}{3}\right)$$

Theory: z_{max} occurs at one of the corners of the bounded region

Simplex Solution: A more general approach and not dependent on the existence of a graph.

Introduce slack variables y_1 , y_2 to convert two inequalities into equalities.

$$\begin{aligned} x_1 + 2x_2 + y_1 &= 6 \\ 2x_1 + x_2 + y_2 &= 8 \end{aligned}$$

Rewrite $z = 3x_1 + 4x_2$ as $z - 3x_1 - 4x_2 = 0$

Form a simplex table, as follows:

	non Basic	Variables	Basic	Variables	EB	
z	x_1	x_2	y_1	y_2	1	Sum
0	1	2	1	0	6	10
0	2	1	0	1	8	12
1	-3	-4	0	0	0	-6

Non Basic Variables: x_1 and x_2
Basic Variables: y_1 and y_2
East Block is EB column

To find a solution: Let non basic variables = 0 and then solve for the basic variables.

Initial solution: $(x_1\ ,\ x_2\ ,\ y_1\ ,\ y_2) = (0\ ,\ 0\ ,\ 6\ ,\ 8)\ ;\ z = 0$

This is a feasible solution, since all variables ≥ 0.

Pivot Choice Procedure:

1. Choose a negative number in the bottom row (z row) - this becomes the pivot column (PC).

2. For all positive numbers in the PC, form quotients of EB element divided by PC element, - select the minimum quotient (Note: the quotient could $= 0$ if EB element $= 0$). - this becomes the pivot row (PR).
   Assume -4 is selected in the bottom row and the "-4" column becomes the PC. The quotients are $6/2 = 3$, $8/1 = 8$; so 2 is the selected pivot and we get:

z	x_1	x_2	y_1	y_2	1	Sum
0	1	2	1	0	6	10
0	3	0	-1	2	10	14
2	-2	0	4	0	24	28

2^{nd} feasible solution
$(x_1\,,\,x_2\,,\,y_1\,,\,y_2) = (0\,,\,3\,,\,0\,,\,5)$
$z = \frac{24}{2} = 12$; z has increased

   but is not yet optimal, as long as a negative remains in the "z" row (bottom row)

3. Select "-2" in the bottom row for PC. (It is the only negative there!)
   Quotients: $6/1$ $10/3$; clearly $10/3$ is minimal, so 3 is the next pivot and we get:

z	x_1	x_2	y_1	y_2	1	Sum
0	0	3	2	-1	4	8
0	3	0	-1	2	10	14
3	0	0	5	2	46	56

3^{rd} feasible solution
$(x_1\,,\,x_2\,,\,y_1\,,\,y_2) = \left(\frac{10}{3}\,,\,\frac{4}{3}\,,\,0\,,\,0\right)$
$z = \frac{46}{3}$

The maximum for z has been attained since there are no more negatives in the bottom row. - the "$y's$" can now be dropped. - They have served their purpose.

$$\begin{array}{lccccc} \text{Path} & (0\,,\,0) & \rightarrow & (0\,,\,3) & \rightarrow & \left(\frac{10}{3}\,,\,\frac{4}{3}\right) \\ & \uparrow & & \uparrow & & \uparrow \\ & z=0 & & z=12 & & z=\frac{46}{3} \end{array}$$

There are other paths that lead to $z = \frac{46}{3}$.
Assume the selection of "-3" (rather than -4) as PC in step 1.

z	x_1	x_2	y_1	y_2	1	Sum
0	1	2	1	0	6	10
0	$\boxed{2}$	1	0	1	8	12
1	-3	-4	0	0	0	-6

Quotients: $\uparrow$ $8/2\,,\,6/1 \implies$ "2" = pivot
solution: $(0\,,\,0\,,\,6\,,\,8)$; $z = 0$

$\rightarrow$

z	x_1	x_2	y_1	y_2	1	Sum
0	0	$\boxed{3}$	2	-1	4	8
0	2	1	0	1	8	12
2	0	-5	0	3	24	24

Quotients: $\uparrow$ $8/1\,,\,4/3 \implies$ "3" = pivot
solution: $(4\,,\,0\,,\,2\,,\,0)$; $z = 12$

$\rightarrow$

z	x_1	x_2	y_1	y_2	1	Sum
0	0	3	2	-1	4	8
0	3	0	-1	2	10	14
3	0	0	5	2	46	56

$\left(\frac{10}{3}\,,\,\frac{4}{3}\right)$ $\quad z = \frac{46}{3}$

$$\begin{array}{lccccc} \text{Path} & (0\,,\,0) & \rightarrow & (4\,,\,0) & \rightarrow & \left(\frac{10}{3}\,,\,\frac{4}{3}\right) \\ & \uparrow & & \uparrow & & \uparrow \\ & z=0 & & z=12 & & z=\frac{46}{3} \end{array}$$

Problems (Standard Mix) $\rightarrow$ 2 unknowns

1. Maximize $z = 5x_1 + x_2$ subject to $\begin{cases} 4x_1 + 3x_2 & \leq 12 \\ x_1 + 3x_2 & \leq 6 \\ x_1\ ,\ x_2 & \geq 0 \end{cases}$ $\rightarrow \Big(z = 15 \text{ at } (3\,,\,0)\Big)$

2. Maximize $z = 3x_1 + 2x_2$ subject to $\begin{cases} 4x_1 + 3x_2 & \leq 12 \\ x_1 + 3x_2 & \leq 6 \\ x_1\ ,\ x_2 & \geq 0 \end{cases}$ $\rightarrow \Big(z = 9 \text{ at } (3\,,\,0)\Big)$

3. Maximize $z = 2x_1 + 4x_2$ subject to $\begin{cases} 4x_1 + x_2 & \leq 8 \\ 2x_1 + x_2 & \leq 6 \\ x_1\ ,\ x_2 & \geq 0 \end{cases}$ $\rightarrow \Big(z = 24 \text{ at } (0\,,\,6)\Big)$

4. Maximize $z = 4x_1 + 5x_2$ subject to $\begin{cases} 4x_1 + x_2 & \leq 8 \\ 2x_1 + x_2 & \leq 6 \\ x_1\ ,\ x_2 & \geq 0 \end{cases}$ $\rightarrow \Big(z = 30 \text{ at } (0\,,\,6)\Big)$

5. Maximize $z = 4x_1 - x_2$ subject to $\begin{cases} 7x_1 + 2x_2 & \leq 14 \\ -3x_1 + \ \ x_2 & \leq 3 \\ x_1\ ,\ x_2 & \geq 0 \end{cases}$ $\rightarrow \Big(z = 8 \text{ at } (2\,,\,0)\Big)$

6. Maximize $z = x_1 - 3x_2$ subject to $\begin{cases} 8x_1 + 16x_2 & \leq 32 \\ -4x_1 + \ \ 8x_2 & \leq 8 \\ x_1\ ,\ x_2 & \geq 0 \end{cases}$ $\rightarrow \Big(z = 4 \text{ at } (4\,,\,0)\Big)$

   It will be simpler if $8x_1 + 16x_2 \leq 8$ is reduced to $x_1 + 2x_2 \leq 4$ and $-4x_1 + 8x_2 \leq 8$ is reduced to $-x_1 + 2x_2 \leq 2$ before the simplex table is set up.

7. Maximize $z = 3x_1 + 4x_2$ subject to $\begin{cases} -x_1 + \ \ x_2 & \leq 1 \\ 2x_1 + 4x_2 & \leq 12 \\ x_1\ ,\ x_2 & \geq 0 \end{cases}$ $\rightarrow \Big(z = 18 \text{ at } (6\,,\,0)\Big)$

8. Maximize $z = x_1 + 6x_2$ subject to $\begin{cases} 2x_1 - \ \ x_2 & \leq 2 \\ 3x_1 + 5x_2 & \leq 15 \\ x_1\ ,\ x_2 & \geq 0 \end{cases}$ $\rightarrow \Big(z = 18 \text{ at } (0\,,\,3)\Big)$

Simplex " $\geq$ " Constraints
Change all " $\geq$ " constraints to " $\leq$ " by multiplying throughout by -1 .This will produce negatives in EB and the initial solution will be infeasible.

Phase I: to obtain a feasible solution.

z	x_1 x_2	1	Sum
0		5	
0		-2	
0		-3	
1			

-2 and -3 are potential PR's

Assume the negatives in the EB are the -2, -3 in the table. Select the second or third row as the pivot row (PR) (i.e. one of the rows with negatives in the EB). Any negative number in these rows is a potential pivot. Test all the potential pivots to determine the effect on the EB.
The best scenario is that the EB become completely negative (zeros allowed). If any potential pivot will accomplish this, select it as the pivot.
Failing the ideal case, select a potential negative pivot which will produce the most negatives in the EB. After the pivot operations are performed, change all signs in the simplex table (excluding the top variable row). If there are still negatives in the EB, repeat the procedure until a feasible solution is attained.

Phase II: Standard Maximization Procedure.
Note: A feasible solution is not always obtained. This is evident if no more negative pivots (in potential PR's) exist and there are still negatives in the EB. STOP! It is not possible to get out of Phase I.

Mixed Constraints ($\geq$, $\leq$, $=$)
Change $\geq$ to $\leq$ by multiplying by (-1) , resulting in negatives in the EB.

Phase I: To obtain a feasible solution, all slack variable for the equality constraints must become 0 and are then removed from the system. Assume $3x_1 + 4x_2 = 5$ is an equality constraint.

z	x_1	x_2	y_1	y_2	1
0	3	4	1	0	5
0	...	...	0	1	

In the initial solution, $y_1 = 5$ since it is a basic variable - a pivot needs to be chosen in the row so that y_1 becomes non basic and therefore 0 , so y_1 can be removed from the system. Select a pivot which causes the least damage to the EB. If a positive pivot is selected, keep the EB as positive as possible. If a negative pivot is chosen, try to keep the EB as negative as possible. Remember to change all signs in the simplex table (excluding the variable row) following the use of a negative pivot. After all the $y's$ for the equality constraints have been removed from the system, get rid of the negatives in the EB by selecting negative pivots as described in $\geq$ constraints procedure. As in $\geq$ case, your solution may be infeasible. Once a feasible solution is obtained, proceed to Phase II.

Phase II: Standard Maximization Procedure.

Note: In the event that all constraints in a problem are equalities, choose only positive pivots to drive the $y's$ to 0 . If only negative pivots remain, and the $y's$ are not all removed from the system, then a feasible solution will not be obtained. Hence, the infeasible case! Do not continue!