Math - Calculus II REVIEW # 8

(1 a) Do the sequences converge or diverge? (Justify)

(i) $\left\{ \frac{\pi^n}{n^2} \right\}$; (ii) $\left\{ \left(1 + \frac{2}{5n}\right)^n \right\}$; (iii) $\left\{ \frac{e^n}{n!} \right\}$; (iv) $\left\{ (-1)^{n+1} \left(\frac{n+1}{n+2}\right) \right\}$; (v) $\left\{ \cos \frac{n\pi}{2} \right\}$

(1 b) Do the series converges or diverge? (answer: converge, diverge or do not know and give a reason)

(i) $\sum_{n=1}^{\infty} \frac{\pi^n}{n^2}$; (ii) $\sum_{n=1}^{\infty} \left(1 + \frac{2}{5n}\right)^n$; (iii) $\sum_{n=1}^{\infty} \frac{e^n}{n!}$; (iv) $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{n+1}{n+2}\right)$; (v) $\sum_{n=1}^{\infty} \cos \frac{n\pi}{2}$

(2) Do the series converge or diverge ? Justify your answer

(a) $\sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1} 2^n}{3^{n+1}} - \frac{1}{\sqrt{n}} \right]$; (b) $\sum_{n=1}^{\infty} \left(\frac{2^n + 3^n}{5^n} \right)$; (c) $\sum_{n=2}^{\infty} \frac{1}{n (\ln n)^{1/2}}$

(d) $\sum_{n=1}^{\infty} \frac{1}{e^{\sqrt[3]{n}} \; n^{2/3}}$; (e) $\sum_{n=2}^{\infty} \frac{\ln n}{n}$

(3) Find the sum of the series:

(a) $\sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)}$; (b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 3^n + 5^n}{6^n}$

(4) Evaluate:

(a) $\int x \arctan x \, dx$; (b) $\int \frac{\sqrt{4x^2-9}}{x} \, dx$; (c) $\int \frac{x^3 \, dx}{25-x^2}$

(d) $\int \tan^4 \theta \, d\theta$; (e) $\int \frac{\sec^4 \theta}{\sqrt{\tan \theta}} \, d\theta$; (f) $\int \frac{2x^2-3x-6}{x^2-4} \, dx$

(g) $\int \frac{x^2+14x+4}{(x+1)^2 \, (2x-1)} \, dx$; (h) $\int \frac{dx}{(4x^2-9)^{3/2}}$; (i) $\int \frac{dx}{x \sqrt{16-9x^2}}$

(5) Compute:

(a) $\lim_{x \to 0^+} x^2 (\ln x)^2$; (b) $\lim_{x \to \infty} \frac{\ln(x+1)}{\ln(4x-1)}$; (c) $\lim_{x \to 1^+} \left(\frac{x}{\ln x} - \frac{1}{\ln x} \right)$

(d) $\lim_{x \to +\infty} x \arctan \frac{1}{x}$; (e) $\lim_{x \to 0^+} (1 + 2 \sin x)^{\cot x}$

(6) Evaluate:

(a) $\int_2^{\infty} \frac{\ln x}{x^2} \, dx$; (b) $\int_{-\infty}^{0} \frac{dx}{x^2+1}$; (c) $\int_1^2 \frac{dx}{(x-2)^2}$

Answers:

(1 a) (i) $\frac{\pi^n}{n^2}$ (LH) $\rightarrow \infty$ as $n \rightarrow \infty$ div ; (ii) $\left(1+\frac{2}{5n}\right)^n \rightarrow e^{2/5}$ as $n \rightarrow \infty$ conv

(iii) $\frac{e^n}{n!} \rightarrow 0$ as $n \rightarrow \infty$ conv $\left(\text{Note: } 0 \le \frac{e.e.e....e}{1.2.3....n} \le \frac{e^2}{2}\left(\frac{e}{3}\right)^{n-2} \rightarrow 0 \text{ as } n \rightarrow \infty\right)$

(iv) $(-1)^{n+1}\left(\frac{n+1}{n+2}\right) \begin{cases} 1 \text{ if } n \text{ is odd} \\ -1 \text{ if } n \text{ is even} \end{cases}$ as $n \rightarrow \infty$ div (oscillation)

(v) $\{0, -1, 0, 1, \ldots\}$ div since $\lim_{n\to\infty} \cos\frac{n\pi}{2} = \pm 1, 0$ (D.N.E.) (oscillation)

(1 b) (i) div since $\lim_{n\to\infty} \frac{\pi^n}{n^2} = \infty \ne 0$ (nTT) (ii) div since $\lim_{n\to\infty}\left(1+\frac{2}{5n}\right)^n = e^{2/5} \ne 0$ (nTT)

(iii) nTT fails (later use ratio test) ; (iv) div since $\lim_{n\to\infty}\left|(-1)^{n+1}\left(\frac{n+1}{n+2}\right)\right| = 1 \ne 0$ (nTT)

(v) div since $\lim_{n\to\infty} \cos\frac{n\pi}{2}$ = (D.N.E.) $\ne 0$ (nTT)

(2 a) div $\left(\text{diff of conv G.S. , } r = -\frac{2}{3} \text{ and div p-series}\right)$

(2 b) conv $\left(\text{sum of 2 conv G.S. , } \sum \frac{2^n}{5^n}, r = \frac{2}{5} \text{ and } \sum \frac{3^n}{5^n}, r = \frac{3}{5}\right)$

(2 c) div IT: $f(x) = \frac{1}{x(\ln x)^{1/2}} \ge 0$, cont , $f'(x) = -\frac{1+2\ln x}{2x^2(\ln x)^{3/2}} < 0$ for $x \ge 2$

$$\int_2^\infty \frac{1}{x(\ln x)^{1/2}}\,dx = \lim_{t\to\infty}\left[2\sqrt{\ln x}\right]_2^t = \infty \text{ div}$$

(2 d) conv IT: $f(x) = \frac{1}{e^{\sqrt[3]{x}}\, x^{2/3}} \ge 0$, cont , $f'(x) = -\frac{2+x^{1/3}}{3x^{5/3}\, e^{\sqrt[3]{x}}} < 0$ for $x \ge 1$

$$\int_1^\infty \frac{e^{-\sqrt[3]{x}}}{x^{2/3}}\,dx = \lim_{t\to\infty}\left[-\frac{3}{e^{\sqrt[3]{x}}}\right]_1^t = \frac{3}{e} \text{ conv}$$

(2 e) div IT: $f(x) = \frac{\ln x}{x} \ge 0$, cont , $f'(x) = \frac{1-\ln x}{x^2} < 0$ for $x \ge 3$

$$\int_3^\infty \frac{\ln x}{x}\,dx = \lim_{t\to\infty}\left[\frac{1}{2}(\ln x)^2\right]_3^t = \infty \text{ div}$$

(3 a) $\sum_{n=1}^{\infty} \left(\frac{3}{2n-1} - \frac{3}{2n+1} \right) \rightarrow S_n = 3 - \frac{3}{2n+1} \rightarrow 3$ as $n \rightarrow \infty$

(3 b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, 3^n}{6^n}$ is a G.S. , $r = -\frac{1}{2}$ convergent to $\frac{1}{3}$ and

$\sum_{n=1}^{\infty} \frac{5^n}{6^n}$ is a G.S. , $r = \frac{5}{6}$ convergent to 5 ; sum of given series: $\frac{1}{3} + 5 = \frac{16}{3}$

(4 a) parts and long division: $\frac{x^2}{2} \arctan x - \frac{1}{2}x + \frac{1}{2} \arctan x + C$

(4 b) trig sub: $\sqrt{4x^2-9} - 3 \operatorname{arcsec} \frac{2x}{3} + C$

(4c) alg sub: $-25(25-x^2)^{1/2} + \frac{1}{3}(25-x^2)^{3/2} + C$

(4 d) trig identity: $\frac{1}{3} \tan^3 \theta - \tan \theta + \theta + C$

(4 e) trig identity: $\frac{2}{5} (\tan \theta)^{5/2} + 2 (\tan \theta)^{1/2} + C$

(4 f) long division & partial fractions: $2x - \ln|x-2| - 2 \ln|x+2| + C$

(4 g) partial fractions: $-2 \ln|x+1| - \frac{3}{x+1} + \frac{5}{2} \ln|2x-1| + C$

(4 h) trig sub: $\frac{-x}{9\sqrt{4x^2-9}} + C$; (4 i) trig sub: $\frac{1}{4} \ln \left| \frac{4 - \sqrt{16-9x^2}}{3x} \right| + C$

(5 a) 0 ; (5 b) 1 ; (5 c) 1 ; (5 d) 1 ; (5 e) e^2

(6 a) $\int_2^{\infty} \frac{\ln x}{x^2}\, dx = \lim_{t\to\infty} \left[-\frac{\ln x}{x} - \frac{1}{x} \right]_2^t = \frac{1}{2} (1 + \ln 2)$ (conv)

(6 b) $\int_{-\infty}^{0} \frac{dx}{x^2+1} = \lim_{t\to-\infty} (\arctan x) \Big|_t^0 = \frac{\pi}{2}$ (conv)

(6 c) $\int_1^2 \frac{dx}{(x-2)^2} = \lim_{t\to 2^-} \left[\frac{-1}{x-2} \right]_1^t = +\infty$ (div)