SOLUTIONS TO FALL 2009 LINEAR ALGEBRA (NYC) FINAL EXAM

1. (a) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$, where $s, t \in \mathbf{R}$

   (b) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$, where $s, t \in \mathbf{R}$

2. (a) $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & -1 \end{bmatrix}$

   (b) No, not every column of A is a pivot column (columns of A are linearly dependent.)

   (c) Yes, A has a pivot position in every row (columns of A span $\mathbf{R}$)

3. (a) Never (a homogeneous system)

   (b) $k \neq 0$ and $k \neq 4$

   (c) $k = 0$ or $k = 4$

4. $p(x) = -2 + 2x + x^2$

5. $A^{-1} = \begin{bmatrix} 1 & 2 & -3 \\ -1 & 1 & -1 \\ 0 & -2 & 3 \end{bmatrix}$

6. $L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$

   $U = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

7. $A^{-1} = \begin{bmatrix} O & N^{-1} \\ M^{-1} & -M^{-1}N^{-1} \end{bmatrix}$

8. $(AA^T)^{-1} = \begin{bmatrix} 3 & 10 & 10 \\ 10 & 34 & 33 \\ 10 & 33 & 34 \end{bmatrix}$

9. (a) 320

   (b) 25

   (c) 25

   (d) $-\frac{5}{2}$

10. $B^{-1} = CA$

11. It is given that

$$A^T = -A$$

Therefore

$$|A^T| = |-A|$$

Since, $|A^T| = |A|$ and $|-A| = (-1)^9|A|$ for a 9×9, we can rewrite this statement:

$$|A| = -|A|$$

Thus

$$|A| + |A| = 0$$

$$2|A| = 0$$

$$|A| = 0$$

The same result is not true for 10×10 A, since in that case $|-A| = (-1)^{10}|A| = |A|$

12. (a) $x_3 = -\frac{1}{3}$
    (b) $A\mathbf{x} = \mathbf{0}$ has a unique solution since $|A| \neq 0$.

13. (a) True. Every elementary matrix is invertible, so $|E_1| \neq 0$ and $|E_2| \neq 0$. So, $|E_1E_2| = |E_1||E_2| \neq 0$.
    (b) False. $(A+B)(A-B) = A^2 - AB + BA - B^2$, and AB is not generally equal to BA. (You could also easily find a counterexample with two very simple–but not TOO simple–2×2 matrices.)
    (c) False. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ for a counterexample (but almost any other matrix A will work.)
    (d) True. Let S, T be transformations from $\mathbf{R}^2$ to $\mathbf{R}^2$. Then $S(\mathbf{x}) = A\mathbf{x}$ where $|A| \neq 0$ since S is onto. And $T(\mathbf{x}) = B\mathbf{x}$ where $|B| = 0$ since T is not onto. And $S \circ T(\mathbf{x}) = AB\mathbf{x}$, where $|AB| = |A||B| = |A| \cdot 0 = 0$. Thus $S \circ T$ is not onto.
    (e) False. For example, let $A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Columns 2 and 4 are non-pivot columns, but the form a linearly independent set.

14. The answer is (a)

15. (a) $9 - 4 = 5$
    (b) 4
    (c) 4
    (d) $7 - 4 = 3$

16. (a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix} \right\}$ dim(Col(A))= 3

(b) $\begin{bmatrix} 3 \\ 0 \\ 1 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 6 \\ -1 \\ 1 \\ 3 \end{bmatrix} = - \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}$

(c) $\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}$

(d) $\{ \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 4 \end{bmatrix} \}$

(e) No, there is a row of zeros in R, so there is not a pivot position in every row of A.

17. (a) $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$, dim($S$)= 3

(b) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$, dim($S$)= 2

(c) $\mathcal{B} = \{x, x^2, x^3\}$, dim($S$)= 3

18. (a) Yes

(b) No. Multiplying most vectors in S by -1 will result in a vector not in S.

(c) Yes.

(d) No, since closure under multiplication fails.

19. (a) Not a subspace of $M_{3\times 3}$ since it is not closed under addition. For example, let $\mathbf{u} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then $\mathbf{u}, \mathbf{v} \in S$, but $\mathbf{u} + \mathbf{v} \neq S$.

(b) Yes, all three axioms hold. (The student needs to confirm.)

20. (a) $\mathcal{B} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

(b) $(-2, -6, -5)$

(c) $\frac{5}{3}$

(d) $\mathbf{x} = t \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$, where $t \in \mathbf{R}$.

21. (a) $\frac{3\sqrt{3}}{2}$
    (b) $x + y + z = 6$
    (c) $3/\sqrt{2}$

22. $-\mathbf{j}$

23. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

24. **Proof.**

Since $\{v_1, v_2, v_3\}$ is linearly dependent, there exist c_1, c_2, c_3 not all zero such that $c_1v_1 + c_2v_2 + c_3v_3 = \mathbf{0}$. Taking T of both sides, we get

$$T(c_1v_1 + c_2v_2 + c_3v_3) = \mathbf{0}$$

The definition of a linear transformation allows us to distribute the T on the left. Also, $T(\mathbf{0}) = \mathbf{0}$ for all linear transformations. This yields:

$$T(c_1v_1) + T(c_2v_2) + T(c_3v_3) = \mathbf{0}$$

and

$$c_1T(v_1) + c_2T(v_2) + c_3T(v_3) = \mathbf{0}$$

But since c_1, c_2, c_3 are not all zero, this means that $\{T(v_1), T(v_2), T(v_3)\}$ is a linearly dependent set.

25. **Proof.**

Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be linearly dependent.

Consider the equation

$$c_1(\mathbf{v}_1 + \mathbf{v}_2) + c_2(\mathbf{v}_2 + \mathbf{v}_3) + c_3(\mathbf{v}_1 + \mathbf{v}_3) = \mathbf{0}$$

It will suffice to show that c_1, c_2, c_3 must all be zero. Rearranging terms, we get:

$$(c_1 + c_3)\mathbf{v}_1 + (c_1 + c_2)\mathbf{v}_2 + (c_2 + c_3)\mathbf{v}_3 = \mathbf{0}$$

Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, the weights must all be zero.

$c_1 + c_3 = 0$

$c_1 + c_2 = 0$

$c_2 + c_3 = 0$

This system is easily solved (by row reduction, for example) to find the unique solution $c_1 = c_2 = c_3 = 0$. Thus

$\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_2 + \mathbf{v}_3, \mathbf{v}_1 + \mathbf{v}_3\}$ is linearly independent.