(Marks)

(4) 1. Solve the following system of equations: $\begin{cases} 2x_1 - 4x_2 - 2x_3 + 8x_4 = -4 \\ -3x_1 + 4x_2 - x_3 - 2x_4 = 0 \\ -x_1 + 3x_2 + 3x_3 - 9x_4 = 5 \end{cases}$

(4) 2. Set up but DO NOT SOLVE a system of equations which would allow you to find the equation for the degree 2 polynomial passing through the points $(2, 1)$ and $(-1, 11)$ and which has a slope of 3 when $x = 2$.

(6) 3. Given that $A = \begin{bmatrix} 2 & -3 & -9 \\ 1 & -2 & -7 \\ -3 & 5 & 17 \end{bmatrix}$

(a) Find the inverse of A.

(b) Use the adjoint formula and the fact that $|A| = -1$ to find the adjoint of A.

(5) 4. Given the following matrix: $A = \begin{bmatrix} 3 & 5 & 0 & 4 \\ -1 & 3 & 1 & -2 \\ 0 & k & 0 & 1 \\ 0 & 4 & 0 & k \end{bmatrix}$

(a) Find $|A|$ in terms of k.

(b) For what values of k is A non-invertible?

(5) 5. Let A be a 4×4 matrix with $|A| = -3$. Let B be a 4×4 non invertible matrix.

For each part, either provide an answer or write "not enough information".

(a) What the value of $|2A|$?

(b) What is the value of $|AB|$?

(c) What is the value of $|A + B + I|$?

(d) What is the value of $|(A^T A)^{-1}|$?

(e) If M is the reduced row echelon form of A, what is the value of $|M|$?

(6) 6. Given the following matrix: $A = \begin{bmatrix} 3 & 2 & 6 \\ 9 & 4 & 22 \\ -12 & -12 & -11 \end{bmatrix}$

(a) Write A as the product of a lower triangular matrix L and an upper triangular matrix U.

(b) Find elementary matrices E_1, E_2 and E_3 such that $E_3 E_2 E_1 A = U$.

(4) 7. Let A and B be $n \times n$ matrices such that AB is its own inverse i.e. $(AB)^{-1} = AB$.

(a) Which of the following is the inverse of BAB (circle your answer)?

(i) ABA (ii) AB (iii) A (iv) BAB (v) BA (vi) B

(b) Is the matrix B necessarily invertible? Justify your answer.

(c) Evaluate and simplify $(AB + I)(AB + I)$.

(d) What is $(AB + I)^{28}$?

(8) 8. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3z \\ 2x - y \end{pmatrix}$ and let L be the line $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

(a) Find the standard matrix for the transformation T.

(b) Sketch the image of the line L under T.

(c) Is T 1-1? onto? Justify your answer in each case.

(d) Let L_0 be the line defined by $\begin{bmatrix} 3 \\ 4 \\ b \end{bmatrix} + t \begin{bmatrix} -2 \\ a \\ 2 \end{bmatrix}$. For what a, b does L_0 define the same line as L.

(Marks)

(4) 9. Use Cramer's Rule to find x_3 only in the following system: $\begin{cases} 5x_1 - x_2 - 2x_3 = 7 \\ -4x_1 - 4x_2 + 7x_3 = 4 \\ -x_1 - x_2 + 2x_3 = 1 \end{cases}$

(4) 10. Fill in each blank with the appropriate word. In each case, the appropriate word is either *must*, *might* or *cannot*. No justification is required.

(a) If A is an $n \times n$ matrix such that $\det(A) = 0$, then the system of equations $A\vec{x} = \vec{0}$ ____________ have a solution.

(b) If B is a set of three linearly independent vectors in P_2 (the vector space of all polynomials of degree less than or equal to 2) then B ____________ be a basis for P_2.

(c) If $\vec{u}$ and $\vec{v}$ are vectors in a vector space S then $3\vec{u} - 5\vec{v}$ ____________ also be a vector in S.

(d) If a transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is onto, then there ____________ be a non-zero vector $\vec{x}$ such that $T(\vec{x}) = \vec{0}$.

(8) 11. Find a specific example of each of the following:

(a) A 3×3 matrix with every entry different such that $|A| = 0$.

(b) Two orthogonal vectors in R^3 that have no zero entries.

(c) Two 2×2 matrices $A \neq 0$ and $B \neq 0$ such that $AB = 0$.

(d) A 2 dimensional subspace of the vector space P_2.

(7) 12. Given that $A = \begin{bmatrix} 2 & 4 & 20 & 7 & 0 & 20 & 17 \\ 2 & -4 & -4 & -11 & -12 & -12 & -21 \\ 1 & 0 & 4 & -1 & -3 & 2 & -1 \\ -2 & 3 & 1 & 6 & 5 & -3 & 8 \end{bmatrix}$ row reduces to

$$R = \begin{bmatrix} 1 & 0 & 4 & 0 & -1 & 6 & 2 \\ 0 & 1 & 3 & 0 & -3 & -5 & -2 \\ 0 & 0 & 0 & 1 & 2 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a basis for the column space of A.

(b) Find a basis for the row space of A.

(c) Find a basis for the null space of A.

(d) What is $\text{rank}(A)$?

(e) What is $\dim(\text{Nul}(A))$?

(f) What is $\text{rank}(A^T)$?

(g) What is $\dim(\text{Nul}(A^T))$?

(6) 13. Let $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : |x| = |y| \right\}$ be a subset of $\mathbf{R}^2$.

(a) Does H contain the zero vector of $\mathbf{R}^2$? Justify.

(b) Is H closed under vector addition? Justify.

(c) Is H closed under scalar multiplication? Justify.

(d) Is H a vector subspace of $\mathbf{R}^2$? Justify.

(5) 14. Let H be the set of all 2×2 matrices such that the sum of the entries in H is 0.

(a) Give an example of an invertible matrix which belongs to H.

(b) Find a basis for this subspace of $M_{2\times 2}$.

(c) What is the dimension of H?

(3) 15. Find the point of intersection between the plane $3x - 2y + 5z = 3$ and the line $\mathbf{x} = \begin{bmatrix} -2 \\ -4 \\ 8 \end{bmatrix} + t \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}$.

(Marks)

(9) 16. Let P_1 be the plane $2x + 3y + 3z = -8$.

Let P_2 be the plane $x + 2y + 2z = -6$.

Let P_3 be the plane $x + 2y + 2z = 1$.

(a) Find the equation of the line of intersection between P_1 and P_2.

(b) What is the cosine of the angle between P_1 and P_2?

(c) Find the distance from P_2 to P_3.

(6) 17. Let P be the plane containing the points $Q(1, 2, 3), R(2, 3, 3)$ and the origin $O(0, 0, 0)$.

Let S be the point $S(6, 4, -2)$.

(a) Find a normal vector to P.

(b) Find an equation for the plane P (in standard form $ax + by + cz = d$).

(c) Find an equation for the plane parallel to P through the point S (also in standard form).

(6) 18. Let $S = \{\vec{v_1}, \vec{v_2}, ..., \vec{v_n}\}$ be a set of linearly independent vectors in a vector space V.

(a) Define what it means for the vectors in S to be linearly independent.

(b) Suppose $T : V \to W$ is a 1-1 linear transformation. Prove that the set $T(S) = \{T(\vec{v_1}), T(\vec{v_2}), ..., T(\vec{v_n})\}$ is also linearly independent.

Answers: 1. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 6 \\ 5 \\ 0 \\ 1 \end{bmatrix}$ 2. $a_0 + 2a_1 + 4a_2 = 1$, $a_0 - a_1 + a_2 = 11$, $a_1 + 4a_2 = 3$ 3. a) $A^{-1} = \begin{bmatrix} -1 & -6 & -3 \\ -4 & -7 & -5 \\ 1 & 1 & 1 \end{bmatrix}$ b) adj(A)=$(-1)A^{-1}$ 4. a) $|A| = -3(k^2 - 4)$ b) $k = 2, k = -2$ 5. a) -48 b) 0 c) not enough info d) $\frac{1}{9}$ e) 1 6. a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 6 \\ 0 & -2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$ b) $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ 7. a) $(BAB)^{-1} = A$ b) Yes, since $(ABA)B = I$. c) $2(AB + I)$ d) $2^{27}(AB + I)$ 8. a) $A = \begin{bmatrix} 0 & 0 & 3 \\ 2 & -1 & 0 \end{bmatrix}$ b) The line is $x + 3y - 3 = 0$. c) It is not 1-1 but it is onto. d) $a = -2$, $b = -1$ 9. $x_3 = 0$ 10. a) might b) must c) must d) might 11. Answers will vary. 12. a) $\left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ -4 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ -11 \\ -1 \\ 6 \end{bmatrix} \right\}$ b) $\{(1, 0, 4, 0, -1, 6, 2), (0, 1, 3, 0, -3, -5, -2), (0, 0, 0, 1, 2, 4, 3)\}$ c) $\left\{ \begin{bmatrix} -4 \\ -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 5 \\ 0 \\ -4 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 0 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ d) 3 e) 4 f) 3 g) 1 13. a) Yes b) No c) Yes d) No 14. a) Answers will vary b) $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \right\}$ c) 3 15. $(4, 2, -1)$ 16. a) $\begin{bmatrix} 2 \\ -4 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ b) $\frac{14}{3\sqrt{22}}$ c) $\frac{21}{9}$ 17. a) $(-3, 3, -1)$ b) $-3x + 3y - z = 0$ c) $-3x + 3y - z + 4 = 0$