(Marks)

(6) 1. Let $f(x) = \int_0^x t\cos\sqrt{t}\,dt$:

(a) find a power series representation for $f(x)$;

(b) use this series to approximate $f(x) = \int_0^{1/2} t\cos\sqrt{t}\,dt$ correctly to 4 decimal places.

(6) 2. Find the power series representation for each of the following functions, and state the radius of convergence.

(a) $f(x) = \dfrac{1}{4-3x}$, centered at $x = 2$.

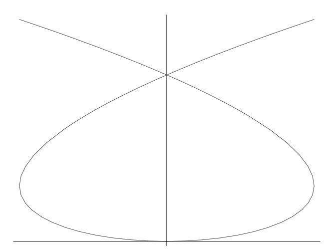
(b) $f(x) = \dfrac{3}{2+x-x^2}$, centered at $x = 0$.

(8) 3. Let $f(x) = \sqrt[3]{8+x}$:

(a) use the Binomial theorem to find the first 5 terms of the Maclaurin series for $f(x)$, and its radius of convergence;

(b) approximate $\sqrt[3]{8.2}$ correctly to 4 decimal places.

(8) 4. Let $\mathcal{C}$ be the plane curve defined by parametric equations $\begin{cases} x = 3t - t^3 \\ y = 3t^2 \end{cases}$



(a) Show the orientation of $\mathcal{C}$.

(b) Find and simplify $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

(c) At what points does $\mathcal{C}$ have a vertical tangent line?

(d) Set up (*but do not evaluate*) the integral needed to find the area of the region enclosed by the loop.

(6) 5. (a) Sketch the graph of $r = 2\sin(3\theta)$.

(b) Find the area of the region enclosed by the curve.

(c) Set up (*but do not evaluate*) the integral needed to find the length of one loop of the curve.

(10) 6. Let $\mathcal{C}$ be the space curve defined by the vector equation $\boldsymbol{r}(t) = \langle e^t, e^t\sin t, e^t\cos t\rangle$.

(a) Find the equation of a quadric surface on which $\mathcal{C}$ lies. Sketch both the surface and the curve.

(b) Find the unit tangent vector $\boldsymbol{T}$ and the unit normal vector $\boldsymbol{N}$.

(c) Find the length of $\mathcal{C}$ on the interval $0 \le t \le 1$.

(d) Find the curvature κ of $\mathcal{C}$.

(e) Find the parametric equations of the tangent line to $\mathcal{C}$ at the point where $t = 0$.

(9) 7. Sketch and describe the following. Show all your work.

(a) The surface $f(x,y) = \sqrt{x^2+2y^2+1}$.

(b) The level curve of $z = \dfrac{y}{x^2+y^2}$ corresponding to $z = \frac{1}{4}$.

(c) The surface $\rho = \csc\varphi\cot\varphi$.

(Marks)

(2) 8. Let $\boldsymbol{r}$ be a three-times-differentiable function of t. Simplify: $[\boldsymbol{r}\cdot(\boldsymbol{r}'\times\boldsymbol{r}'')]'$.

(4) 9. Find the limit (or if appropriate, show that it does not exist):

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}}$ (b) $\displaystyle\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$

(3) 10. Show that if $f(t)$ is differentiable, then $z = f(x/y)$ is a solution of the partial differentiable equation $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 0$.

(3) 11. Let $\mathcal{C}$ be the curve formed by the intersection of the level surface $x^2y + yz + z^2 + 1 = 0$ and the plane $x + y + z = 1$. Let $P_0(1, -1, 1)$ be a point on $\mathcal{C}$. Find a tangent vector to $\mathcal{C}$ at P_0.

(6) 12. Let $z = f(x, y)$ be implicitly defined by $\sin(xy) + xz^4 + y^3z = 2$, and let $P_0(0, 1, 2)$ be a point on this surface.

(a) Find the equation of the tangent plane to the surface at P_0.

(b) Find $\nabla f(0, 1)$.

(c) Find an approximation of $f(-0.05, 1.10)$.

(5) 13. Find and classify the critical points of $f(x, y) = y^2 + x^2y + x^2 - 2y$.

(5) 14. Use Lagrange Multipliers to find the points on the sphere $x^2 + y^2 + z^2 = 3$ where the maximum and minimum values of the product xyz are found.

(8) 15. Evaluate:

(a) $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2+y^2+1)\,dx\,dy$ (b) $\displaystyle\int_0^4\int_0^1\int_{2y}^{2} \frac{\cos(x^2)}{\sqrt{z}}\,dx\,dy\,dz$

(5) 16. Sketch the solid region $\mathcal{S}$ bounded below by $z = \sqrt{x^2+y^2}$, and bounded above by $\rho = 2\cos\phi$. Find the volume of $\mathcal{S}$.

(6) 17. Sketch the solid region $\mathcal{S}$ bounded below by the plane $z = 0$, laterally by the surface $x^2 + (y-1)^2 = 1$, and above by the surface $z = x^2 + y^2$.
Set up the triple integrals representing the volume of $\mathcal{S}$ in

(a) cartesian coordinates (b) cylindrical coordinates

1. (a) $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+2}}{(n+2)(2n)!}$

   (b) $f(\frac{1}{2}) \simeq \frac{1}{8} - \frac{1}{48} + \frac{1}{1536} \simeq 0.1048$

2. (a) $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^n (x-2)^n}{2^{n+1}}$, and $R = \frac{2}{3}$

   (b) $f(x) = \sum_{n=0}^{\infty} \left(\frac{1}{2^{n+1}} + (-1)^n \right) x^n$, and $R = 1$

3. (a) $f(x) = 2 + \frac{x}{12} - \frac{x^2}{288} + \frac{5x^3}{20736} - \frac{5x^4}{248832} + \cdots$ and $R = 8$

   (b) $f(0.2) = 2 + \frac{0.2}{12} - \frac{(0.2)^2}{288} \simeq 2.0165$

   with absolute value of error less than $\frac{5(0.2)^3}{20736} = 0.19 \times 10^{-5}$

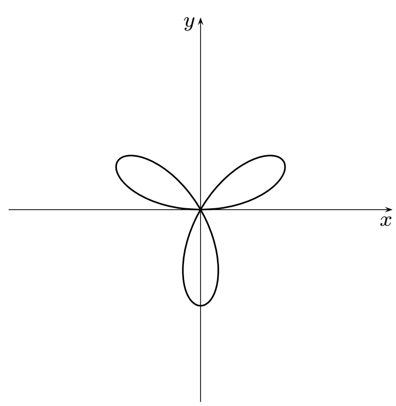
4. (a) Counterclockwise orientation

   (b) $\frac{dy}{dx} = \frac{2t}{1-t^2}$ and $\frac{d^2y}{dx^2} = \frac{2(1+t^2)}{3(1-t^2)^3}$

   (c) Vertical tangents at $(\pm 2, 3)$

   (d) $A = 2\int_0^{\sqrt{3}} x dy = 2\int_0^{\sqrt{3}} (3t - t^3)(6t)dt = 12\int_0^{\sqrt{3}} (3t^2 - t^4)dt$

5. (a)



   (b) $A = \int_0^{\pi} \frac{1}{2}(2\sin(3\theta))^2 d\theta = \pi$

   (c) $\mathcal{L} = \int_0^{\pi/3} \sqrt{4 + 32\cos^2(3\theta)} d\theta = 2\int_0^{\pi/3} \sqrt{1 + 8\cos^2(3\theta)} d\theta$

6. (a) The curve lies on the cone $x^2 = y^2 + z^2$. Note that $x = e^t$ so $x > 0$ implying that the curve spirals around $x = \sqrt{y^2 + z^2}$, the upper nappe of the cone.

   (b) $\mathbf{T}(t) = \frac{1}{\sqrt{3}} \langle 1, \sin t + \cos t, \cos t - \sin t \rangle$

   and $\mathbf{N}(t) = \frac{1}{\sqrt{2}} \langle 0, \cos t - \sin t, -\sin t - \cos t \rangle$

   (c) $\mathcal{L} = \sqrt{3}(e - 1)$

(d) $\kappa = \frac{\sqrt{2}}{3e^t}$

(e) $x = 1 + t,\ y = t,\ z = 1 + t$ where $t \in \mathbb{R}$

7. (a) $-x^2 - 2y^2 + z^2 = 1$ and $z > 0$, hyperboloid of two sheets (top part only)

   (b) $x^2 + (y-2)^2 = 4$, circle of radius 2 and center $(0, 2)$

   (c) $z = r^2$ or $z = x^2 + y^2$, circular paraboloid

8.

$$\begin{aligned}[\mathbf{r} \cdot (\mathbf{r}' \times \mathbf{r}'')]' &= \mathbf{r}' \cdot (\mathbf{r}' \times \mathbf{r}'') + \mathbf{r} \cdot (\mathbf{r}' \times \mathbf{r}'')' \\ &= 0 + \mathbf{r} \cdot (\mathbf{r}'' \times \mathbf{r}'' + \mathbf{r}' \times \mathbf{r}''') \\ &= \mathbf{r} \cdot (\mathbf{r}' \times \mathbf{r}''')\end{aligned}$$

9. (a) The limit does not exist

   (b) Use polar coordinates to show the limit is zero

10. Show that $\frac{\partial z}{\partial x} = \frac{df}{dt}\left(\frac{1}{y}\right)$ and $\frac{\partial z}{\partial y} = \frac{df}{dt}\left(\frac{-x}{y^2}\right)$.

11. Let $F(x, y, z) = x^2y + yz + z^2 + 1$ and $\mathbf{n} = \langle 1, 1, 1\rangle$. Then $\mathbf{n} \times \nabla F(P_0)$ gives $\mathbf{v} = \langle -1, -3, 4\rangle$

12. (a) $17x + 6y + z = 8$

    (b) $\nabla f(0, 1) = \langle -17, -6\rangle$

    (c) $f(-0.05, 1.1) \simeq f(0, 1) + df|_{(0,1)} = 2.25$

13. There is a local minimum at $(0, 1)$ ($f(0, 1) = -1$). The points $(\pm 2, -1)$ are saddle points.

14. The minimum value is -1 occurring at $(-1, -1, -1)$, $(-1, 1, 1)$, $(1, -1, 1)$ and $(1, 1, -1)$.
    The maximum value is 1 occurring at $(-1, -1, 1)$, $(-1, 1, -1)$, $(1, -1, -1)$ and $(1, 1, 1)$.

15. (a) $I = \int_0^{2\pi} \int_0^1 \ln(r^2 + 1) r\,dr\,d\theta = \pi(2 \ln 2 - 1)$

    (b) $I = \int_0^4 \int_0^2 \int_0^{\frac{x}{2}} \frac{\cos(x^2)}{\sqrt{z}} dy\,dx\,dz = \sin 4$

16. $V = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{2\cos\phi} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \pi$

17. (a) $V = \int_0^2 \int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} \int_0^{x^2+y^2} dz\,dx\,dy$

    (b) $V = \int_0^{\pi} \int_0^{2\sin\theta} \int_0^{r^2} r\,dz\,dr\,d\theta$