1. (5 points) You are given the following matrix A and vector $\mathbf{b}$.

$$A = \begin{bmatrix} 1 & 3 & 2 & 1 \\ 2 & 6 & 3 & 2 \\ 3 & 9 & -1 & -1 \\ -4 & -12 & 4 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 6 \\ 11 \\ 3 \\ -2 \end{bmatrix}$$

(a) Is $\begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \end{bmatrix}$ is a solution to the $A\mathbf{x} = \mathbf{b}$?

(b) Find the general solution to the system $A\mathbf{x} = \mathbf{b}$.

2. (6 points) Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

(a) Is it true that for each $\mathbf{b}$ in $\mathbb{R}^3$ the system $A\mathbf{x} = \mathbf{b}$ is consistent? Justify your answer.

(b) Evaluate AA^T.

(c) True or False (Justify.)

i. A and AA^T have the same column space.

ii. A and AA^T have the same null space.

iii. $\text{Row}(A)$ and $\text{Row}(AA^T)$ are not the same, but they have the same dimension.

3. (4 points) Find the values of a, b and c in the quadratic polynomial $f(x) = ax^2 + bx + c$ if $f(-1) = -4$, $f(2) = 5$ and $f(3) = 0$.

4. (4 points) Given $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 4 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 15 \\ 4 \end{bmatrix}$.

Solve the system $A\mathbf{x} = \mathbf{b}$ using the given LU factorization.

5. (8 points) Let $A = \begin{bmatrix} 2 & -2 & 3 \\ 0 & 1 & 2 \\ 1 & -3 & -3 \end{bmatrix}$.

(a) Find A^{-1}.

(b) Consider the 5×5 matrix $U = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 2 & -2 & 3 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 1 & -3 & -3 & 0 & 0 \end{bmatrix}$, which may also be viewed as the block matrix $\begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$. Use your answer from part (a) and this block matrix to find U^{-1}.

6. (6 points) Let A, B, and C be 3×3 matrices. It is given that $\det(A) = 3$ and $\det(B) = 7$. It is also given that C is noninvertible. For each of the following expressions, either evaluate or write "not enough information".

(a) $\det(10A^2B^{-1})$

(b) $\det(A + B)$

(c) $\det(AC + BC)$

7. (5 points) Let $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & -4 & 3 \\ 3 & -2 & 1 \end{bmatrix}$

(a) Find $\det(A)$.

(b) Given $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, solve for x_2 only, using Cramer's Rule.

8. (7 points) Consider the set $S = \{A : A \in M_{3\times3} \text{ and } \operatorname{rank}(A) \leqslant 2\}$. (Justify all answers.)

(a) Give an example of a non-zero matrix in S.

(b) Does S contain the zero matrix?

(c) Is S closed under addition?

(d) Is S closed under scalar multiplication?

(e) Is S a subspace of $M_{3\times3}$?

9. (6 points) You are given the vector space

$V = \{p(x) : p \in \mathbb{P}_3 \text{ and } p'(0) = 0 \text{ and } p''(0) = 0\}$

Find a basis for V.

10. (6 points) Use the words MIGHT, CANNOT, or MUST to complete the statements below, as appropriate:

(a) If the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ spans a plane in $\mathbb{R}^3$, then the set $\{\mathbf{u}, \mathbf{v}\}$ ____________ span the same plane.

(b) If the column vectors of a square matrix A span all of $\mathbb{R}^3$, then the determinant of A ____________ be zero.

(c) The columns of an $n \times n$ elementary matrix ____________ be a basis for $\mathbb{R}^n$.

(d) If $A \in M_{5\times6}$, then $\operatorname{rank}(A)$ ____________ be equal to $\operatorname{rank}(A^T)$

(e) For any invertible matrix A, the rank of A ____________ be the same as the rank of A^2.

(f) If a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ given by $T(\mathbf{x}) = A\mathbf{x}$ is onto, then A ____________ be invertible.

11. (6 points) Let $\mathcal{L}$ be the line $\mathbf{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(a) Find a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ is a rotation that transforms $\mathcal{L}$ into a horizontal line.

(b) Find a non-zero matrix B such that $T(\mathbf{x}) = B\mathbf{x}$ transforms $\mathcal{L}$ into a single point.

12. (11 points) Let $\mathcal{L}_1$ be the line $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ and let $\mathcal{L}_2$ be the line $\mathbf{x} = \begin{bmatrix} 13 \\ 14 \\ 0 \end{bmatrix} + s \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

(a) Find the point of intersection of $\mathcal{L}_1$ and $\mathcal{L}_2$.

(b) Find an equation of the form $ax + by + cz = d$ for the plane containing $\mathcal{L}_1$ and $\mathcal{L}_2$.

(c) Find the distance from $\mathcal{L}_1$ to the origin.

13. (8 points) Given $A(-2, 1, 0)$, $B(1, 5, 0)$, $C(4, 0, 1)$.

(a) Find the area of the triangle $\triangle ABC$.

(b) Find an equation for the line through A that is perpendicular to the plane containing A, B, and C.

(c) Find a unit vector parallel to $\overrightarrow{AB}$.

(d) Find a point between A and B that is two units away from A.

14. (7 points) Let $\mathcal{P}_1$ be the plane $3x + y - 4z = 10$ and $\mathcal{P}_2$ be the plane $-x + 3y - z = 5$.

(a) Find the cosine of the angle between $\mathcal{P}_1$ and $\mathcal{P}_2$.

(b) Find an equation for the line through the origin that is parallel to both $\mathcal{P}_1$ and $\mathcal{P}_2$.

15. (3 points) Suppose $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$ such that $\|\mathbf{u}\| = 3$, $\|\mathbf{v}\| = 5$, and $\|\mathbf{u}+\mathbf{v}\| = 7$. Find the angle between $\mathbf{u}$ and $\mathbf{v}$. Hint: start by looking at $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v})$.

16. (4 points) Let $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$. Show that if $\text{Proj}_{\mathbf{v}}\mathbf{u} = \text{Proj}_{\mathbf{v}}\mathbf{w}$, then $\mathbf{u} - \mathbf{w}$ is perpendicular to $\mathbf{v}$.

17. (4 points) Given $T : M_{n\times n} \to M_{n\times n}$ defined by $T(X) = AXA^{-1}$.

(a) Show that $\det(T(X)) = \det(X)$.

(b) Show that $T(X)T(Y) = T(XY)$

Answers

1. (a) Yes, (b) $\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, where $t \in \mathbb{R}$

2. (a) Yes, because A has a pivot in every row, (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

   (c) i. True: $\text{Col}(A) = \text{Col}(AA^T) = \mathbb{R}^3$

   ii. False: $\dim \text{Nul}(A) = 2, \dim \text{Nul}(AA^T) = 0$

   iii. True: $\dim \text{Row}(A) = \dim \text{Row}(AA^T) = 3$,
   but $\text{Row}(A) \subseteq \mathbb{R}^5$ and $\text{Row}(AA^3) \subseteq \mathbb{R}^3$

3. $a = -2, b = 5, c = 3$

4. $\mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

5. $A^{-1} = \begin{bmatrix} -3 & 15 & 7 \\ -2 & 9 & 4 \\ 1 & -4 & -2 \end{bmatrix}$

   $U^{-1} = \begin{bmatrix} 0 & 0 & -3 & 15 & 7 \\ 0 & 0 & -2 & 9 & 4 \\ 0 & 0 & 1 & -4 & -2 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$

6. (a) $\frac{9000}{7}$, (b) not enough information, (c) 0

7. (a) 15, (b) $\frac{7}{15}$

8. (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, (b) Yes, since the rank of a zero matrix is 0. (c) No, since $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are both in S, but their sum is not. (d) Yes. S is equal to the set of all 3×3 matrices with determinant 0. If $\det(A) = 0$, then $= \det(kA) = k^3 \det(A) = k^3 \cdot 0 = 0$. (e) No, since S is not closed under addition.

9. $\mathcal{B} = \{1, x^3\}$

10. (a) might, (b) cannot, (c) must, (d) must, (e) must, (f) must

11. (a) $A = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$, (b) $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

12. (a) $(7, 10, -2)$, (b) $x - y - z = -1$, (c) $\frac{\sqrt{91}}{7}$

13. (a) $\frac{\sqrt{754}}{2}$, (b) $\mathbf{x} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ 3 \\ 27 \end{bmatrix}$ where $t \in \mathbb{R}$, (c) $\begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \\ 0 \end{bmatrix}$, (3) $\begin{bmatrix} -\frac{4}{5} \\ \frac{13}{5} \\ 0 \end{bmatrix}$

14. (a) $\frac{4}{\sqrt{286}}$, (b) $\mathbf{x} = t \begin{bmatrix} 11 \\ 7 \\ 10 \end{bmatrix}$ where $t \in \mathbb{R}$

15. $\frac{\pi}{3}$

16. Let $\text{Proj}_{\mathbf{v}}\mathbf{u} = \text{Proj}_{\mathbf{v}}\mathbf{w}$. We need to show that $(\mathbf{u} - \mathbf{w}) \cdot \mathbf{v} = 0$. If $\mathbf{v} = \mathbf{0}$, then we are done. If $\mathbf{v} \neq \mathbf{0}$, we write $\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \left(\frac{\mathbf{w}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}$ and conclude that $\mathbf{u} \cdot \mathbf{v} = \mathbf{w} \cdot \mathbf{v}$. So, $\mathbf{u} \cdot \mathbf{v} - \mathbf{w} \cdot \mathbf{v} = 0$, and finally $(\mathbf{u} - \mathbf{w}) \cdot \mathbf{v} = 0$.

17. The definition of T tells us that A is invertible.

    (a) $\det(T(X)) = \det(AXA^{-1}) = \det(A)\det(X)\det(A^{-1}) = \frac{\det(A)}{\det(A)}\det(X) = \det(X)$.

    (b) $T(X)T(Y) = (AXA^{-1})(AYA^{-1}) = AX(A^{-1}A)YA^{-1} = AX(I)YA^{-1} = AXYA^{-1} = T(XY)$