1. (30 points) Evaluate the following integrals.

(a) $\displaystyle\int \frac{x^3+4}{x^2(x+2)^2}\,dx$ (b) $\displaystyle\int \frac{\ln x}{x^{3/2}}\,dx$

(c) $\displaystyle\int x\sec^6(x^2)\,dx$ (d) $\displaystyle\int \frac{dx}{(4x^2+1)^{5/2}}$

(e) $\displaystyle\int_0^1 x\arctan x\,dx$

(f) $\displaystyle\int_0^{\pi/3} \sec x\ln(\sec x+\tan x)\,dx$

2. (6 points) Evaluate the following limits.

(a) $\displaystyle\lim_{x\to 1^-}\frac{\arccos x}{\sqrt{1-x}}$ (b) $\displaystyle\lim_{x\to 0}(e^x-x)^{1/x^2}$

3. (10 points) Evaluate each improper integral or show it diverges.

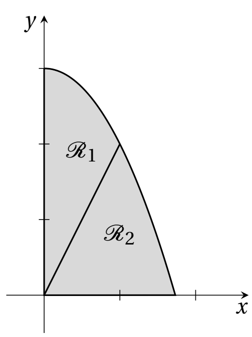
(a) $\displaystyle\int_1^2 \frac{dx}{\sqrt{4-x^2}}$ (b) $\displaystyle\int_{-\infty}^{\infty}\frac{e^x}{e^{2x}+1}\,dx$

4. (3 points) Set up, but **do not evaluate**, the integral(s) for the area of the region enclosed by the parabola $y=x^2$ and the curve $y=x^3-12x$.

5. (6 points) The shaded area in the figure is bounded by $y=3-x^2$, $x=0$, and $y=0$. The line $y=2x$ subdivides this area into two regions $\mathscr{R}_1$ and $\mathscr{R}_2$. Set up, but **do not evaluate**, an integral for the volume of the solid obtained by:

(a) rotating $\mathscr{R}_1$ about the y-axis,

(b) rotating $\mathscr{R}_2$ about the vertical line $x=-1$.



6. (5 points) Express y as a function of x if

$$\sqrt{x^2+9}\,\frac{dy}{dx}=xe^{4-y}$$

and $y=4$ if $x=0$.

7. (5 points) A chemical plant discharges toxic solvents into the ground at a rate of 5 tons per year. These solvents do not all stay in the ground: each year, $\frac{1}{10}$ of the total amount of solvents evaporates into the air.

(a) Find a formula for the total amount $A(t)$ of solvents in the ground after t years, assuming there are initially none.

(b) In the long run, how many tons of solvents will accumulate in the ground?

8. (4 points) Determine whether the sequence $\{a_n\}$ converges or diverges. Justify your answer: if the sequence converges, find the limit; otherwise, explain why it diverges.

(a) $a_n=\dfrac{\cos(n!)}{5n+1}$ (b) $a_n=(-1)^n\dfrac{e^n}{e^n+n}$

9. (4 points) Given the series $\displaystyle\sum_{n=1}^{\infty}\ln\left(\frac{2n-1}{2n+1}\right)$,

(a) find an expression for its partial sums s_n,

(b) use $\{s_n\}$ to determine whether the series is convergent or divergent. If it is convergent, find its sum.

10. (9 points) Determine whether the series converges or diverges. Justify your answer.

(a) $\displaystyle\sum_{n=1}^{\infty} n\sin\left(\frac{1}{n}\right)$ (b) $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{2n+3}}{5n^2-4n}$

(c) $\displaystyle\sum_{n=1}^{\infty}\left[\frac{\ln n}{\sqrt{n}}-\frac{1}{6^n}\right]$

11. (8 points) Determine whether the series is absolutely convergent, conditionally convergent, or divergent. Justify your answer.

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\left(\frac{2^{n+1}}{n+2^n}\right)^{-n}$

(b) $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{n\sqrt{\ln n}}$

12. (5 points) Find the radius and interval of convergence of the power series

$$\sum_{n=0}^{\infty}\frac{3^n(x+1)^n}{\sqrt{2n+1}}$$

13. (2 points) Write the first four terms of the Maclaurin series for the function $f(x)=(x+1)e^{2x}$ given that

$$f'(x)=(2x+3)e^{2x},\quad f''(x)=(4x+8)e^{2x},$$
$$f'''(x)=(8x+20)e^{2x}$$

14. (3 points) Given that the series $\sum_{n=1}^{\infty}a_n$ is absolutely convergent, are the following series convergent or divergent? Briefly justify.

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n a_n$ (b) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{1+a_n^2}$

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{|a_n|}{n}$

ANSWERS

1. (a) $2\ln|x+2|+\dfrac{1}{x+2}-\ln|x|-\dfrac{1}{x}+C$

(b) $-\dfrac{2\ln x+4}{\sqrt{x}}+C$

(c) $\frac{1}{10}\tan^5(x^2)+\frac{1}{3}\tan^3(x^2)+\frac{1}{2}\tan(x^2)+C$

(d) $\dfrac{x}{\sqrt{4x^2+1}}-\dfrac{4x^3}{3(4x^2+1)^{3/2}}+C=\dfrac{8x^3+3x}{3(4x^2+1)^{3/2}}+C$

(e) $\frac{1}{2}(x^2+1)\arctan x-\frac{1}{2}x\Big]_0^1=\frac{1}{4}(\pi-2)$

(f) Letting $u=\ln(\sec x+\tan x)$, the integral equals

$$\tfrac{1}{2}u^2\Big]_0^{\ln(2+\sqrt{3})}=\tfrac{1}{2}\ln^2(2+\sqrt{3})$$

2. (a) $\sqrt{2}$ (b) $\sqrt{e}$

3. (a) Converges to $\frac{1}{3}\pi$ (b) Converges to $\frac{1}{2}\pi$

4. $\displaystyle\int_{-3}^{0}(x^3-12x-x^2)\,dx+\int_0^4(x^2-x^3+12x)\,dx$

5. (a) $\displaystyle\int_0^1 2\pi x(3-x^2-2x)\,dx$

(b) $\displaystyle\int_0^2\pi\Big[\big(\sqrt{3-y}+1\big)^2-\big(\tfrac{1}{2}y+1\big)^2\Big]\,dy$

6. $y=4+\ln\big(\sqrt{x^2+9}-2\big)$

7. (a) $A(t)=50(1-e^{-t/10})$ (b) $\lim\limits_{t\to\infty}A(t)=50\,\text{tons}$

8. (a) Since $-1<\cos(n!)<1$,

$$-\frac{1}{5n+1}<a_n<\frac{1}{5n+1}\quad\text{for all } n,$$

and so $\lim\limits_{n\to\infty}a_n=0$ by the Squeeze Theorem.

(b) Since

$$\lim_{n\to\infty}\frac{e^n}{e^n+n}=1\neq 0,$$

$\{a_n\}$ oscillates between values close to 1 and values close to -1, i.e., $\{a_n\}$ diverges.

9. (a) $s_n=-\ln(2n+1)$

(b) $\lim\limits_{n\to\infty}s_n=-\infty$, so the series diverges (to $-\infty$)

10. Let a_n be the nth term of the series in question.

(a) Diverges by the test for divergence:

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{\sin(1/n)}{1/n}=\lim_{x\to 0}\frac{\sin x}{x}=1\neq 0$$

(b) Since

$$a_n=\frac{\sqrt{n}\sqrt{2+3/n}}{n^2(5-4/n)}=\frac{1}{n^{3/2}}\,\frac{\sqrt{2+3/n}}{5-4/n}$$

use the limit comparison test with the convergent p-series $\sum b_n=\sum 1/n^{3/2}$:

$$\frac{a_n}{b_n}=\frac{\sqrt{2+3/n}}{5-4/n}\longrightarrow\frac{\sqrt{2}}{5}\neq 0,\infty$$

and so $\sum a_n$ converges.

(c) $\sum(\ln n)/\sqrt{n}$ diverges by direct comparison with the divergent p-series $\sum 1/\sqrt{n}$ since

$$\frac{\ln n}{\sqrt{n}}>\frac{1}{\sqrt{n}}\quad\text{for all } n\geqslant 3$$

(or use the integral test). On the other hand, $\sum 1/6^n$ is a geometric series with $r=1/6$, and so it is convergent because $|r|<1$. The given series therefore diverges, since it is the difference of a divergent series and a convergent series.

11. Let a_n be the nth term of the series in question.

(a) Converges absolutely by the root test:

$$|a_n|^{1/n}=\left(\frac{2^{n+1}}{n+2^n}\right)^{-1}=\frac{n+2^n}{2^{n+1}}=\frac{n}{2^{n+1}}+\frac{1}{2}\longrightarrow\frac{1}{2}<1$$

(b) Converges conditionally. $\sum|a_n|$ diverges by the integral test: $f(x)=1/(x\sqrt{\ln x})$ is continuous, positive and decreasing on $[2,\infty)$ and

$$\int_2^\infty f(x)\,dx=\lim_{t\to\infty}2\sqrt{\ln x}\,\Big]_2^t=\infty$$

On the other hand, $1/(n\sqrt{\ln n})\to 0$ as $n\to\infty$ and is decreasing, and so $\sum a_n$ converges by the alternating series test.

12. $R=\frac{1}{3}$, $[-\frac{4}{3},-\frac{2}{3})$

13. $1+3x+4x^2+\frac{10}{3}x^3+\cdots$

14. (a) This series is absolutely convergent (and therefore convergent) because $\sum|(-1)^n a_n|=\sum|a_n|$ converges.

(b) Since $\sum a_n$ is convergent, $\lim_{n\to\infty}a_n=0$, and so

$$\lim_{n\to\infty}a_n^2=0\implies\lim_{n\to\infty}\frac{1}{1+a_n^2}=1\neq 0$$

so the given series diverges by the divergence test.

(c) This series converges by direct comparison with the covergent series $\sum|a_n|$ since

$$\frac{|a_n|}{n}\leqslant|a_n|\qquad\text{for all } n\geqslant 1$$