1. (12 points) Evaluate the following limits.

Use $-\infty$, ∞ or "does not exist", wherever appropriate.

(a) $\displaystyle\lim_{x\to 3} \frac{\frac{3}{7} - \frac{x}{3x-2}}{x-3}$

(b) $\displaystyle\lim_{x\to -3^+} \frac{x^2-9}{x^2+6x+9}$

(c) $\displaystyle\lim_{x\to 0} \frac{x+\sin x}{\tan x}$

(d) $\displaystyle\lim_{x\to 0} \frac{2\cos^2 x - 7\cos x + 5}{\cos x - 1}$

(e) $\displaystyle\lim_{x\to -\infty} \frac{\sqrt{x^2+1} - (7-2x)}{x-2}$

(f) $\displaystyle\lim_{x\to \infty} \frac{2x+\sin x}{3x}$

2. Consider the function

$$f(x) = \begin{cases} \dfrac{x+3}{x^2+7x+12} & \text{if } x < -3, \\ x^2+3 & \text{if } -3 \leqslant x < -2 \text{ and} \\ \dfrac{x+9}{x+3} & \text{if } x > -2. \end{cases}$$

(a) (4 points) Identify all points of discontinuity, and state whether the discontinuity is removable, jump or infinite.

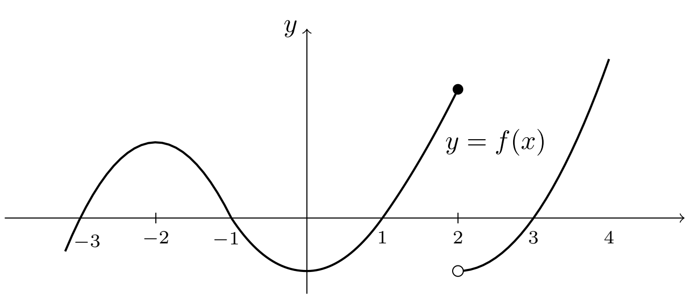
(b) (2 points) Find all horizontal asymptotes of f.

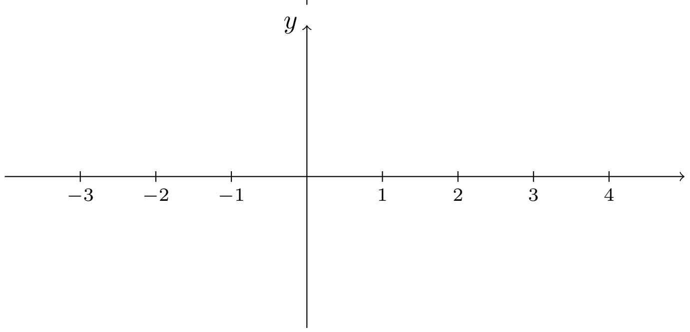
3. Consider the function $f(x) = \sqrt{x^2-5}$.

(a) (3 points) Find $f'(x)$ by using the limit definition of the derivative.

(b) (2 points) Find an equation of the tangent line to the curve $y = f(x)$ at $x = 3$.

4. (2 points) Given the following graph of f, draw a rough sketch of a graph of its derivative f' on the given axes.





5. (15 points) Find the derivative $\dfrac{dy}{dx}$ for each of the following. Do not simplify your answers.

(a) $y = 1 + 2x^3 + \dfrac{4}{\sqrt[5]{x}} + 6^x + \tan\left(\frac{7}{8}\pi\right) + 9\log_{10}(x)$

(b) $y = \sec(3x^2+2)\cos(8xe^x)$

(c) $y = \ln\left(\sqrt[5]{\dfrac{(2x^3+1)\sin x}{(4x-1)^6}}\right)$

(d) $\sqrt{4x^2+3y^3} = x + y$

(e) $y = \dfrac{5x}{1+x^{\sin x}}$ (*Hint: what is* $\frac{d}{dx}(x^{\sin x})$*?*)

6. (2 points) Show that if f, g and h are differentiable, and h is not zero, then $\left(\dfrac{fg}{h}\right)' = \dfrac{f'gh + fg'h - fgh'}{h^2}$.

7. (2 points) Find the 81st derivative of $f(x) = \cos(10x)$.

8. (3 points) For what value(s) of x in the interval $[0, 2\pi]$ does the curve $y = e^x \cos x$ have a horizontal tangent?

9. (5 points) Barbara is flying a kite in a large field. The kite is 50 feet above the ground and moves horizontally away from Barbara at a speed of 4 ft/s. At what rate is the angle between the string and the horizontal changing when 100 feet of string have been let out?

10. (4 points) Show that the equation $\cos(3x) + 2016x = 0$ has exactly one solution.

11. (4 points) Find the absolute maximum and minimum values of $f(x) = (x^2+2x)^{2/3}$ on the interval $[-3, 2]$.

12. Given $f(x) = \dfrac{(x-1)^2}{(x+1)^2}$, $f'(x) = \dfrac{4(x-1)}{(x+1)^3}$ and $f''(x) = \dfrac{8(2-x)}{(x+1)^4}$:

(a) (1 point) state the domain of f;

(b) (2 points) find the intervals on which f is increasing and decreasing;

(c) (1 point) find all local maxima and minima of f;

(d) (2 points) find the intervals on which f is concave up and concave down;

(e) (1 point) find any points of inflection of f.

13. (5 points) You are given the following information about a function f:

- f is continuous everywhere except at -1.
- x-intercepts: $-2, 1, 3$
- y-intercept: -1
- $f(2) = 2,\ f(4) = 2$
- f has a vertical asymptote at $x = -1$
- $\lim_{x\to-\infty} f(x) = -3$ $\lim_{x\to\infty} f(x) = 4$
- $f'(x) > 0$ on $(-\infty, -1) \cup (-1, 2) \cup (3, \infty)$
- $f'(x) < 0$ on $(2, 3)$
- $f''(x) > 0$ on $(-\infty, -1) \cup (1, 2) \cup (2, 4)$
- $f''(x) < 0$ on $(-1, 1) \cup 4, \infty)$

Use all of the above information to sketch a graph of f. Clearly label all asymptotes, local extrema and points of inflection.

14. (5 points) Find the point(s) on the curve $y = x^2$ that are closest to the point $(0, 3)$. Justify your answer.

15. (3 points) Find the function f satisfying

$f''(x) = x^2 + \sin x - 2e^x + 3,\ f'(0) = 1$ and $f(0) = 4$.

16. (12 points) Evaluate the following integrals.

(a) $\displaystyle\int \left(\frac{\sqrt{x^5}}{x^3} - e^x - \cos 3\right) dx$

(b) $\displaystyle\int_1^e \frac{x^2 + 2x + 1}{x^3 + x^2}\, dx$

(c) $\displaystyle\int (\sec^2 x)(1 + \sin x)\, dx$

(d) $\displaystyle\int_0^4 |2x - 3|\, dx$

17. (2 points) Evaluate $\displaystyle\lim_{n\to\infty} \frac{1}{n}\left(\frac{1^5 + 2^5 + 3^5 + \cdots + n^5}{n^5}\right)$ by expressing it as a definite integral.

18. Given

$$g(x) = \int_2^{\sqrt{x}} \frac{t}{\ln(1+t)}\, dt,$$

find:

(a) (1 point) $g(4)$

(b) (2 points) $g'(x)$

19. (2 points) Given that

$$\int_1^3 f(x)\, dx = 8, \qquad \int_2^5 f(x)\, dx = -3,$$

$$\text{and} \int_1^2 f(x)\, dx = \int_2^3 f(x)\, dx,$$

find:

(a) $\displaystyle\int_2^3 f(x)\, dx;$

(b) $\displaystyle\int_1^5 f(x)\, dx.$

20. (1 point) If $\lim_{x\to4} f(x) = 0$ and $\displaystyle\lim_{x\to4} \frac{f(x)}{g(x) - \pi} = 10$, then what is $\lim_{x\to4} g(x)$?

Answers

1. (a) $\frac{2}{49}$

(b) $-\infty$

(c) 2

(d) -3

(e) 1

(f) $\frac{2}{3}$

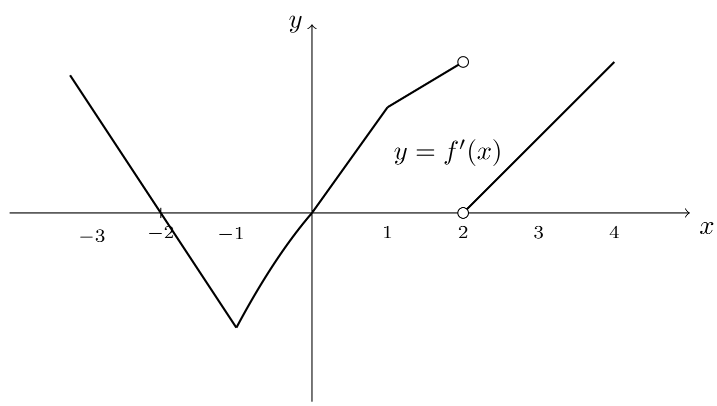
2. (a) $x = -4$ (infinite), $x = -3$ (jump), $x = -2$ (removable)

(b) $y = 0$ (at $-\infty$) and $y = 1$ (at $+\infty$)

3. (a) $f'(x) = \dfrac{x}{\sqrt{x^2 - 5}}$

(b) $y - 2 = \frac{3}{2}(x - 3)$

4.



5. (a) $y' = 6x^2 - \frac{4}{5}x^{-6/5} + 6^x \ln 6 + \dfrac{9}{x \ln 10}$

(b) $y' = 6x \sec\left(3x^2+1\right)\tan\left(3x^2+1\right)\cos\left(8xe^x\right) - \dfrac{8e^x\,(x+1)\sin\left(8xe^x\right)}{\cos\left(3x^2+2\right)}$

(c) $y' = \dfrac{1}{5}\left(\dfrac{6x^2}{2x^3+1} + \cot x - \dfrac{24}{4x-1}\right)$

(d) $y' = -\dfrac{8x - 2(x+y)}{9y^2 - 2(x+y)}$ (Simplified)

(e) $y' = \dfrac{5\left(1+x^{\sin x}\right) - 5x \cdot x^{\sin x}\left(\ln x \cos x + \dfrac{\sin x}{x}\right)}{\left(1+x^{\sin x}\right)^2}$

6.

$$\left(\frac{fg}{h}\right)' = \frac{(fg)'\,h - fgh'}{h^2} = \frac{(f'g+fg')\,h - fgh'}{h^2} = \frac{f'gh + fg'h - fgh'}{h^2}$$

7. As $f^{(4)}(x) = 10^4 \cos 10x$, we have that $f^{(80)}(x) = 10^{80}\cos 10x$ and so $f^{(81)}(x) = -10^{81}\sin 10x$

8. $f'(x) = e^x\,(\cos x - \sin x)$, so $f'(x) = 0$ if $\sin x = \cos x$, which happens at $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$ in the interval $[0, 2\pi]$

9. If Barbara is x feet from the point on the ground directly beneath the kite and θ is the angle of elevation of the string, then $x = 50\cot\theta$, so $\dfrac{dx}{dt} = -50\csc^2\theta\,\dfrac{d\theta}{dt}$, or equivalently, $\dfrac{d\theta}{dt} = -\dfrac{1}{50}\sin^2\theta\,\dfrac{dx}{dt}$

Now $\dfrac{dx}{dt} = 4$, and when 100 feet of string have been released, $\sin\theta = \frac{50}{100} = \frac{1}{2}$, so at this instant the angle of elevation of the string is decreasing at a rate of $\frac{1}{50}\cdot\left(\frac{1}{2}\right)^2\cdot 4 = \frac{1}{50}$ radians per second.

10. Let $f(x) = \cos(3x) + 2016x$. This function is continuous and differentiable everywhere.

In particular, the intermediate value theorem will apply to any closed interval. Letting $x = 0$, $f(0) = 1$, and letting $x = -1$, $f(-1) = \cos(-3) - 2016$. Clearly $-2017 \le f(-1) \le -2015$, so $f(-1)$ is negative. Applying the IVT on the interval $[-1, 0]$, we find that f must have a root in this interval, that is, there must be a number c, with $-1 < c < 0$, such that $f(c) = 0$, since $f(-1) < 0 < f(0)$.

Suppose now that there are *two* roots c and d, and suppose (without loss of generality) that $c < d$. Then as $f(c) = f(d) = 0$, and f is continuous and differentiable everywhere, we can apply Rolle's theorem on the interval $[c, d]$. We find that there must be a number r in the interval (c, d) such that $f'(r) = 0$.

However, computing f', we find that $f'(x) = -3\sin(3x) + 2016 > 0$ for any x. Therefore, by contradiction, there cannot be two roots f.

This shows that c is the only root of f.

11. $f'(x) = \dfrac{4(x+1)}{3\left(x^2+2x\right)^{1/3}}$. The critical numbers of f are $x = -1$, $x = 0$ and $x = -2$

Calculating the value of f and the critical numbers and the endpoints, we find:

$f(-3) = \sqrt[3]{9}$, $f(-2) = 0$, $f(-1) = 1$, $f(0) = 0$ and $f(2) = 4$

Thus the absolute maximum value is 4, attained at $x = 2$, and the absolute minimum value is 0, attained at $x = -2$ and $x = 0$.

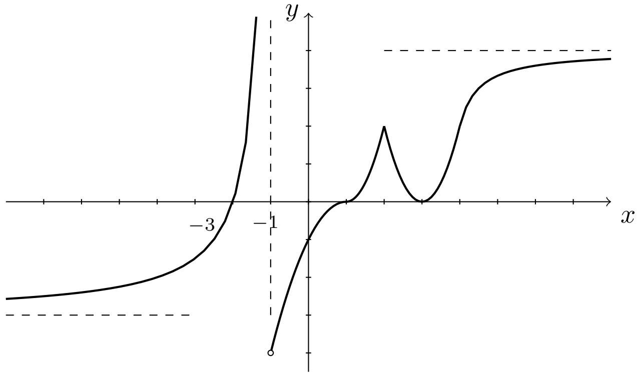
12. (a) $\mathbb{R}\backslash\{-1\}$, or $(-\infty, -1) \cup (-1, \infty)$

(b) f is increasing on $(-\infty, -1)$ and $(1, \infty)$ and decreasing on $(-1, 1)$

(c) Local minimum at $x = 1$, at the point $(1, 0)$

(d) f is CU on $(-\infty, -1)$ and $(-1, 2)$ and CD on $(2, \infty)$

(e) There is one inflection point at $(2, \frac{1}{9})$

13. 

14. The points are $\left(\pm\sqrt{\frac{5}{2}}, \frac{5}{2}\right)$ with minimum distance $\frac{\sqrt{11}}{2}$

15. $f(x) = \frac{1}{12}x^4 - \sin x - 2e^x + \frac{3}{2}x^2 + 4x + 6$

16. (a) $2\sqrt{x} - e^x - (\cos 3)\, x + C$

(b) $2 - \frac{1}{e}$

(c) $\tan x + \sec x + C$

(d) $\frac{17}{2}$

17. This limit is $\displaystyle\int_0^1 x^5\, dx = \frac{1}{6}$

18. (a) $g(4) = 0$

(b) $g'(x) = \dfrac{1}{2\ln\left(1+\sqrt{x}\right)}$

19. (a) $\displaystyle\int_2^3 f(x)\, dx = 4$

(b) $\displaystyle\int_1^5 f(x)\, dx = 1$

20. If $\lim_{x\to 4} g(x) \neq \pi$, then $\displaystyle\lim_{x\to 4} \frac{f(x)}{g(x)-\pi} = 0$. Since this limit is given to be 10, it must be that $\lim_{x\to 4} g(x) = \pi$