Solutions

1. a. If $y=\sqrt{\sin x}$ then $2y\,dy=\cos x\,dx$ and $\cos^2 x = 1-y^4$, so

$$\int \frac{\cos^3 x}{\sqrt{\sin x}}\,dx = 2\int (1-y^4)\,dy = 2\left(y-\tfrac{1}{5}y^5\right)+a = \tfrac{2}{5}(5-\sin^2 x)\sqrt{\sin x}+a.$$

b. If $y=\arcsin(x^2)$ then $dy = 2x(1-x^4)^{-1/2}\,dx$, so

$$\int \frac{x\arcsin(x^2)}{\sqrt{1-x^4}}\,dx = \tfrac{1}{2}\int y\,dy = \tfrac{1}{4}\left(\arcsin(x^2)\right)^2+b.$$

c. The resolution into partial fractions of the integrand is

$$\frac{x+6}{x(x^2+2x+3)} = \frac{2}{x} - \frac{2x+3}{x^2+2x+3},$$

where the coefficient over x is found by inspection (covering and evaluating) and the coefficients over x^2+2x+3 are obtained by comparing the quadratic and linear terms of the numerator. The integral of the second partial fraction is

$$\int\left\{\frac{2x+2}{x^2+2x+3}+\frac{1}{(x+1)^2+2}\right\}dx = \log(x^2+2x+3)+\tfrac{1}{2}\sqrt{2}\arctan\left(\tfrac{1}{2}\sqrt{2}(x+1)\right)+c,$$

and therefore

$$\int\frac{x+6}{x(x^2+2x+3)}\,dx = \log\frac{x^2}{x^2+2x+3}-\tfrac{1}{2}\sqrt{2}\arctan\left(\tfrac{1}{2}\sqrt{2}(x+1)\right)+c.$$

d. Repeated partial integration and absorption gives

$$\begin{aligned}\int \sin(\log x)\,dx &= x\sin(\log x) - x\cos(\log x) - \int\sin(\log x)\,dx\\ &= \tfrac{1}{2}x\left(\sin(\log x)-\cos(\log x)\right)+d.\end{aligned}$$

e. If $y=\sqrt{x^2-4}$, then $y^2=x^2-4$, so $y\,dy = x\,dx$, or $dx/(xy) = dy/(x^2)$, and thus

$$d\left(\frac{y}{x^2}\right) = \frac{dy}{x^2}-\frac{2y\,dx}{x^3} = \frac{dy}{x^2}-\frac{2(x^2-4)}{x^3y}\,dx = \frac{8\,dx}{x^3y}-\frac{dy}{y^2+4}.$$

Therefore,

$$\int\frac{dx}{x^3\sqrt{x^2-4}} = \frac{y}{8x^2}+\frac{1}{8}\int\frac{dy}{y^2+4} = \frac{\sqrt{x^2-4}}{8x^2}+\frac{1}{16}\arctan\left(\tfrac{1}{2}\sqrt{x^2-4}\right)+E.$$

f. Multiplying and dividing by $\sqrt{3+x}$ omits -3 from the domain of the integrand, and gives

$$\int\frac{3+x}{\sqrt{9-x^2}}\,dx = 3\arcsin\left(\tfrac{1}{3}x\right)-\int dy = 3\arcsin\left(\tfrac{1}{3}x\right)-\sqrt{9-x^2}+f,$$

where in the second term $y=\sqrt{9-x^2}$, so that $dy = -(x/y)\,dx$.

2. a. Revising the expression in the limit and using the fact that $\lim_{\vartheta\to0}\frac{\sin\vartheta}{\vartheta}=1$ gives

$$\lim_{x\to0+}\frac{\log(\sin x)}{\log(\sin 2x)} = \lim_{x\to0+}\frac{1+\log\left(\frac{\sin x}{x}\right)/(\log x)}{1+\log\left(2\frac{\sin 2x}{2x}\right)/(\log x)} = 1.$$

b. If $y=\frac{1}{2}\pi - x$ and $z=1/y$ then

$$\lim_{x\to\frac{1}{2}\pi-}(\tan x)^{2x-\pi} = \lim_{y\to0+}(\cot y)^{-2y} = \lim_{y\to0+}\left(\frac{\sin y}{y\cos y}\right)^{2y}\cdot\lim_{z\to\infty}e^{-2(\log z)/z}=1$$

by elementary properties of the logarithm (the definition of $\log z$ implies that $0<\log z<z$ if $z>1$, which immediately gives $0<(\log z)^a/z^b<(2a/b)^a z^{-b/2}$ for $z>1$ and $a,b>0$).

c. Combining terms and using the Maclaurin expansion of the exponential function gives

$$\lim_{x\to0}\frac{e^{3x}-1-3x}{x(e^{3x}-1)} = \lim_{x\to0}\frac{\frac{9}{2}+\frac{9}{2}x+\frac{27}{8}x^2+\cdots}{3+\frac{9}{2}x+\frac{9}{2}x^2+\cdots} = \frac{3}{2}.$$

(Alternatively, two applications of l'Hôpital's rule could be used.)

3. a. Partial integration gives

$$\int_0^\infty(-xe^{-x})\,dx = \lim_{t\to\infty}(x+1)e^{-x}\Big|_0^\infty = \lim_{t\to\infty}\left\{\frac{t+1}{e^t}-1\right\}=-1,$$

via basic properties of the exponential function (the inequality proved in Part c of Question 2 immediately gives $0<y^a/e^{by^c}<(2a/(bc))^{a/c}e^{-\frac{1}{2}by^c}$ for $a,b,c,y>0$).

b. Integrating by inspection gives

$$\int_0^2\frac{dx}{(x-1)^{2/3}} = \lim_{s\to1-}3\sqrt[3]{x-1}\Big|_0^s+\lim_{t\to1+}3\sqrt[3]{x-1}\Big|_t^2 = 6.$$

4. For $-\frac{1}{2}\pi<x<\frac{1}{2}\pi$, the equation in question is equivalent to

$$\frac{1}{\sqrt{y^2+4}}\frac{dy}{dx}=\tan x,\qquad\text{or}\qquad \log\left(y+\sqrt{y^2+4}\right)=\log(\sec x)+C,$$

which is equivalent to $y+\sqrt{y^2+4}=A\sec x$ (in which $A=e^C$). The initial condition $y=0$ if $x=0$ gives $A=2$, so $y+\sqrt{y^2+4}=2\sec x$. To express y as a function of x, observe that subtracting y and squaring gives $4\sec^2 x-4y\sec x=4$, or $y=\sec x-\cos x=\sin x\tan x$.

5. If $\overline{y}=x^3+x+4$ and $\underline{y}=x^3+x^2+3x+1$ then $\overline{y}-\underline{y}=-x^2-2x+3=(3+x)(1-x)$, which is positive if $-3<x<1$ and vanishes if x is -3 or 1. So the area of the region enclosed by the curves is

$$\int_{-3}^1(\overline{y}-\underline{y})\,dx = \int_{-3}^1(-x^2-2x+3)\,dx = \left(-\tfrac{1}{3}x^3-x^2+3x\right)\Big|_{-3}^1 = -\tfrac{28}{3}+8+12=\tfrac{32}{3}.$$

6. If $\overline{y}=x+1$ and $\underline{y}=x^3+x$ then $\overline{y}-\underline{y}=1-x^3$, which is positive if $0<x<1$ and vanishes if $x=1$. The solid obtained by revolving $\mathcal{R}$ about the x axis consists of annuli of inner radius x^3+x and outer radius $x+1$, for $0\leqslant x\leqslant1$, so its volume is equal to

$$\pi\int_0^1\left\{(x+1)^2-(x^3+x)^2\right\}dx.$$

The solid obtained by revolving $\mathcal{R}$ about the line defined by $x=3$ consists of concentric cylindrical shells of radius $3-x$ and height $1-x^3$, for $0\leqslant x\leqslant1$, so its volume is equal to

$$2\pi\int_0^1(3-x)(1-x^3)\,dx.$$

7. If $x=\frac{1}{4}y^2-\frac{1}{2}\log y$, then

$$1+\left(\frac{dx}{dy}\right)^2 = 1+\left(\tfrac{1}{2}y-\tfrac{1}{2}y^{-1}\right)^2 = \tfrac{1}{4}y^2+\tfrac{1}{2}+\tfrac{1}{4}y^{-2} = \left(\tfrac{1}{2}y+\tfrac{1}{2}y^{-1}\right)^2,$$

and hence

$$\int_1^y\sqrt{1+\left(\frac{dx}{d\eta}\right)^2}\,d\eta = \tfrac{1}{2}\int_1^y\left(\eta+\eta^{-1}\right)d\eta = \tfrac{1}{4}(y^2-1)+\tfrac{1}{2}\log y,$$

which is the length of the curve between $\left(\frac{1}{4},1\right)$ and (x,y) if $y\geqslant1$, and is -1 times the length of the curve between $\left(\frac{1}{4},1\right)$ and (x,y) if $0<y<1$.

8. a. Since
$$\lim_{t\to 0}(1+t)^{1/t}=e, \qquad \lim_{n\to\infty}\frac{2n}{3n-1}=\tfrac{2}{3}$$
and
$$a_n=\left(\frac{3n+1}{3n-1}\right)^n=\left(1+\frac{2}{3n-1}\right)^{\frac{3n-1}{2}\cdot\frac{2n}{3n-1}},$$
it follows that $\lim_{n\to\infty} a_n=e^{2/3}$.

b. Since
$$a_n=\frac{n^3(2n)!}{(2n+2)!}=\frac{n^3}{(2n+2)(2n+1)}=\frac{n}{2(1+1/n)(2+1/n)},$$
it follows that the sequence $\{a_n\}$ diverges to ∞.

9. If
$$a_n=\frac{1}{n}-\frac{1}{n+2} \qquad \text{and} \qquad A_n=\frac{1}{n}+\frac{1}{n+1},$$
then $a_n=A_n-A_{n+1}$ for $n\geqslant 1$, and the sum of the first n terms of the series is
$$a_1+a_2+\cdots+a_n=A_1-A_{n+1}=\tfrac{3}{2}-\frac{1}{n+1}-\frac{1}{n+2}.$$
Hence, the sum of the series is $\lim_{n\to\infty}(a_1+a_2+\cdots+a_n)=\tfrac{3}{2}$.

10. a. If $n\geqslant 1$ then
$$a_n=\frac{\sqrt{n^2+3}}{3n^2+7}>\frac{\sqrt{n^2}}{3n^2+7n^2}=\frac{1}{10n}>0,$$
so the comparison test implies that $\sum a_n$ diverges with the harmonic series.

b. Since $\frac{d}{dx}(x^{-1/4}\log x)=\frac{1}{4}x^{-5/4}(4-\log x)$ is positive if $0<x<e^4$ and negative if $x>e^4$, it follows that
$$0\leqslant a_n=\frac{\log n}{n^{3/2}}=\frac{\log n}{n^{1/4}}\cdot\frac{1}{n^{5/4}}<\frac{4}{e}\cdot\frac{1}{n^{5/4}},$$
for $n\geqslant 1$. Therefore, the comparison test implies that $\sum a_n$ converges with the p-series $\sum n^{-5/4}$.

11. a. Since $\sum 2^{-n}$ is a convergent geometric series, and
$$0<a_n=\frac{n!}{1\cdot 3\cdot 5\cdot 7\cdots(2n+1)}=1\cdot\frac{1}{3}\cdot\frac{2}{5}\cdot\frac{3}{7}\cdots\frac{n}{2n+1}<\left(\frac{1}{2}\right)^n,$$
for $n\geqslant 1$, the comparison test implies that $\sum(-1)^n a_n$ is absolutely convergent.

b. If $n>3$ then
$$a_n=\frac{n^n}{3^{n+1}}=\frac{1}{3}\cdot\frac{n}{3}\cdot\frac{n}{3}\cdot\frac{n}{3}\cdots\frac{n}{3}>\frac{n}{9}>0, \qquad \text{so} \qquad \lim_{n\to\infty} a_n=\infty.$$
Hence, the vanishing condition implies that $\sum(-1)^n a_n$ is divergent.

c. If $n\geqslant 1$ then
$$a_n=\frac{1}{\sqrt{5n+3}}\geqslant\frac{1}{\sqrt{5n+3n}}=\frac{\sqrt{2}}{4}\cdot\frac{1}{\sqrt{n}}>0,$$
so the comparison test implies that $\sum a_n$ diverges with the p-series $\sum n^{-1/2}$. On the other hand,
$$a_n=\frac{1}{\sqrt{5n+3}}>\frac{1}{\sqrt{5n+8}}=a_{n+1}$$
if $n\geqslant 1$, and $\lim a_n=0$, so the alternating series test implies that $\sum(-1)^n a_n$ is convergent. Therefore, the series $\sum\cos(n\pi)a_n=\sum(-1)^n a_n$ is conditionally convergent.

12. If $x\neq -1$ and
$$\alpha_n=\frac{(-1)^n(x+1)^n}{5^n\sqrt{n}}, \qquad \text{then} \qquad \lim\left|\frac{\alpha_{n+1}}{\alpha_n}\right|=\lim\frac{1}{5\sqrt{1+1/n}}|x+1|=\tfrac{1}{5}|x+1|,$$
so the ratio test implies that $\sum\alpha_n$ is absolutely convergent if $|x+1|<5$, *i.e.*, $-6<x<4$, and divergent if $x<-6$ or $x>4$. If $x=-6$ then $\sum\alpha_n=\sum n^{-1/2}$ is a divergent p-series ($p=\frac{1}{2}\leqslant 1$), and if $x=4$ then $\sum\alpha_n=\sum(-1)^n n^{-1/2}$, which is convergent by the alternating series test ($n^{-1/2}>(n+1)^{-1/2}$ if $n\geqslant 1$, and $\lim n^{-1/2}=0$). Therefore, the radius of convergence of $\sum\alpha_n$ is 5, and the interval of convergence of $\sum\alpha_n$ is $(-6,4]$.

13. From the expansion $1/(1+t)=\sum_{k=0}^{\infty}(-1)^k t^k$ (a geometric series), it follows that
$$\begin{aligned}\frac{1}{2+x}&=\frac{1}{3}\cdot\frac{1}{1+\frac{1}{3}(x-1)}=\frac{1}{3}\sum_{k=0}^{\infty}\frac{(-1)^k}{3^k}(x-1)^k\\&=\sum_{k=0}^{\infty}\frac{(-1)^k}{3^{k+1}}(x-1)^k\\&=\tfrac{1}{3}-\tfrac{1}{9}(x-1)+\tfrac{1}{27}(x-1)^2-\tfrac{1}{81}(x-1)^3+\cdots,\end{aligned}$$
which is valid if $\frac{1}{3}|x-1|<1$, or equivalently, $-2<x<4$.