1. Given that $A = \begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 1 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$

[4 pts] (a) Solve the system $A\mathbf{x} = \mathbf{b}$

[1 pt] (b) Write $\mathbf{b}$ as a linear combination of columns of A.

[1 pt] (c) What is $\text{rank}(A)$?

[1 pt] (d) What is the dimension of $\text{Nul}(A^T)$?

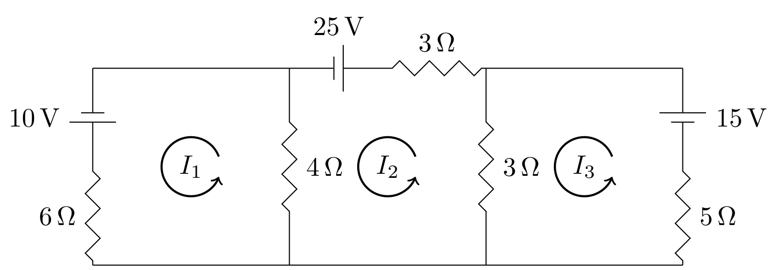
[1 pt] (e) Is $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ in $\text{Nul}(A^T)$? Justify.

2. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 1 \\ k^2-5 \end{bmatrix}$ and $\mathbf{v}_4 = \begin{bmatrix} 2 \\ 3 \\ k \end{bmatrix}$.

[4 pts] (a) For what values of k is $\mathbf{v}_4$ in $\text{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\})$?

[2 pts] (b) For what values of k is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent?

[4 pts] **3.** Set up an augmented matrix for finding the loop currents of the following electrical circuit. Do not solve the system



[5 pts] **4.** Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ -1 & -1 & -3 \end{bmatrix}$.

5. Let $A = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 3 & 12 \\ 1 & 8 & 5 \end{bmatrix}$.

[4 pts] (a) Find an LU-factorization of A.

[4 pts] (b) Write L as a product of elementary matrices.

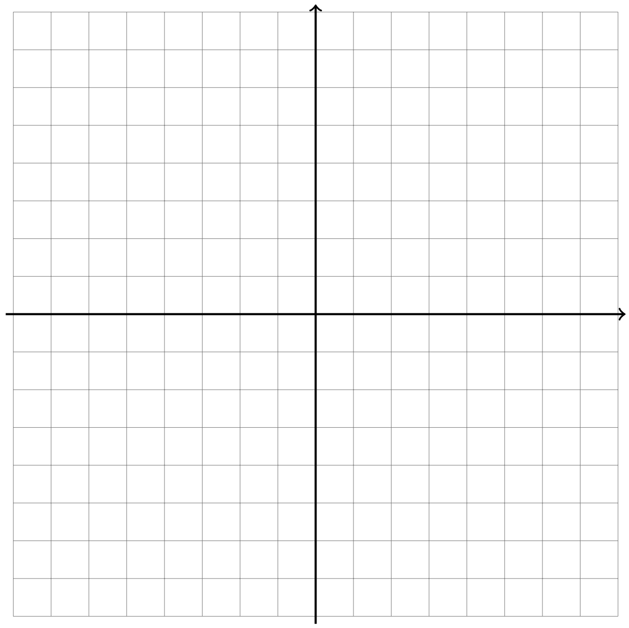
6. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ denote the horizontal expansion (stretching) by a factor of 2, and let $S : \mathbb{R}^2 \to \mathbb{R}^2$ denote the vertical shear that transforms $(1, 0)$ to $(1, 2)$.

[2 pts] (a) Find the standard matrix for S.

[2 pts] (b) Find the standard matrix for T.

[2 pts] (c) Find the standard matrix for the composition $S \circ T$.

[1 pt] (d) Let $\mathcal{R}$ denote the triangle in $\mathbb{R}^2$ whose vertices are $(-2,0), (2,0)$, and $(0,4)$. In the space provided sketch $\mathcal{R}$.



[1 pt] (e) In the same graph sketch the image $(S \circ T)(\mathcal{R})$.

[3 pts] (f) Compute the area of the image $(S \circ T)(\mathcal{R})$.

7. Let $A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$, and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T(\mathbf{x}) = A\mathbf{x}$.

[3 pts] (a) Let $\mathcal{P}$ be the plane given by the parametric-vector equation $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$. Is the image $T(\mathcal{P})$ a plane, a line, or a point? Justify.

[2 pts] (b) Is T one-to-one? Justify.

8. Let A, B, and C be 3×3 matrices. If $\det(A) = 10$, $\det(B) = -2$ and C is non-invertible, evaluate the following determinants. Show your work.

[2 pts] (a) $\det(3B^2A^{-1})$

[2 pts] (b) $\det(C^TA + C^TB)$

[2 pts] (c) $\det((3A)^{-1}B^2)$

9. Given $\det \begin{bmatrix} a & b & c \\ d & e & f \\ 1 & 1 & 1 \end{bmatrix} = 3$, compute the following determinants.

[3 pts] (a) $\det\begin{bmatrix} 2 & 2 & 2 \\ a & b & c \\ d-3 & e-3 & f-3 \end{bmatrix}$

[3 pts] (b) $\det\begin{bmatrix} 0 & 0 & 5 & 10 \\ a & d & 2 & 5 \\ b & e & 2 & 5 \\ c & f & 2 & 5 \end{bmatrix}$

10. Consider the block matrices $M = \begin{bmatrix} I & A \\ A & I \end{bmatrix}$ and $N = \begin{bmatrix} I & 0 \\ -A & I \end{bmatrix}$, where A is an $n \times n$ matrix such that $A^2 = I$.

[2 pts] (a) Compute MN and simplify.

[1 pt] (b) Is M invertible? Justify.

11. Let $\mathbf{u}$ and $\mathbf{v}$ be two unit vectors in $\mathbb{R}^n$ which are orthogonal to each other. Compute the following.

[2 pts] (a) $(2\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - 5\mathbf{v})$

[2 pts] (b) $\|\mathbf{u} + 4\mathbf{v}\|$

[3 pts] **12.** Find the point between the points $P(6, -2, 5)$ and $Q(10, 2, 7)$ whose distance from P is 2 units.

[2 pts] **13.** Find the values of h and k for which the line $\mathbf{x} = \begin{bmatrix} h \\ 2 \\ -1 \end{bmatrix} + t\begin{bmatrix} 1 \\ k \\ 3 \end{bmatrix}$ lies in the plane $x - 2y + z = 5$.

14. You are given the following points: $A(1, 2, 3)$, $B(2, 2, 4)$, and $C(-5, 3, 1)$.

[3 pts] (a) Give a parametric-vector equation for the line containing A and B.

[4 pts] (b) Find the point on the line from part (a) that is closest to the point C.

[3 pts] (c) Find the area of the triangle whose vertices are the points A, B, and C.

15. Let $H = \left\{ \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} : x^2 + y^2 = z^2 \right\}$.

[2 pts] (a) List two matrices that belong to H which are not scalar multiples of each other.

[2 pts] (b) Is H closed under scalar multiplication? Justify.

[2 pts] (c) Is H closed under addition? Justify.

[5 pts] **16.** Given $A = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix}$, consider the subspace W of $\mathbb{R}^3$ given by $W = \{\mathbf{x} \in \mathbb{R}^3 : A\mathbf{x} = -2\mathbf{x}\}$. Find a basis for W.

17. Complete the following statements with "must", "might", or "cannot", as appropriate.

[1 pt] (a) If $T : \mathbb{R}^6 \to \mathbb{R}^8$ is a linear transformation, then T ________ be onto.

[1 pt] (b) If A is row equivalent to B, then $\mathrm{Col}(A)$ ________ equal $\mathrm{Col}(B)$.

[1 pt] (c) If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors and $\mathrm{Proj}_{\mathbf{v}}\,\mathbf{u} = \mathbf{u}$, then $\mathbf{u}$ ________ be parallel to $\mathbf{v}$.

[1 pt] (d) If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in $\mathbb{R}^3$, then $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u}$ ________ be equal to 0.

[2 pts] (e) Let A be a 3×3 matrix, and let B be a 4×4 matrix. If $\mathrm{rank}(A) = \mathrm{rank}(B)$, then $\det(A)$ ________ equal zero and $\det(B)$ ________ equal zero.

[2 pts] **18.** Give an example of a 3×5 matrix $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 & \mathbf{b}_5 \end{bmatrix}$ that satisfies all four of the following conditions.

(i) B is in reduced row echelon form.

(ii) $\mathbf{b}_1$ and $\mathbf{b}_3$ are pivot columns.

(iii) $\{\mathbf{b}_2, \mathbf{b}_4\}$ is a basis for $\mathrm{Col}(B)$.

(iv) $\{\mathbf{b}_2, \mathbf{b}_5\}$ is not a basis for $\mathrm{Col}(B)$.

Answers

1. $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 2 & 0 & 7 \\ 0 & 1 & -2 & -3 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & -5 \end{array}\right]$.

(a) $\mathbf{x} = \begin{bmatrix} 7 \\ -3 \\ 0 \\ 0 \\ -5 \end{bmatrix} + s \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

(b) $\mathbf{b} = 7\,\mathrm{col}_1(A) - 3\,\mathrm{col}_2(A) - 5\,\mathrm{col}_5(A) = 7 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 5 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

(c) $\mathrm{rank}(A) = 3$

(d) $\dim \mathrm{Nul}(A^T) = 3 - \mathrm{rank}(A^T) = 3 - \mathrm{rank}\, A = 0$

(e) No, since $\dim \mathrm{Nul}(A^T) = 0$ implies $\mathrm{Nul}(A^T) = \{\mathbf{0}\}$.
(Alternatively, note $A^T\mathbf{u} = (\mathbf{u}^T A)^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T \neq \mathbf{0}$.)

2. $\left[\begin{array}{ccc|c} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & k^2 - 4 & k - 2 \end{array}\right] = R$

(a) $\mathbf{v}_4 \in \mathrm{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}) \Leftrightarrow \mathrm{col}_4(R)$ is not a pivot column $\Leftrightarrow k \neq -2$

(b) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \Leftrightarrow \mathrm{col}_3(R)$ is a pivot column $\Leftrightarrow k \in \mathbb{R} \setminus \{-2, 2\}$

3. **Loop 1:** $6I_1 + 4(I_1 - I_2) = 10 \Longrightarrow 10I_1 - 4I_2 = 10$

Loop 2: $4(I_2 - I_1) + 3(I_2 - I_3) + 3I_2 = -25 \Longrightarrow -4I_1 + 10I_2 - 3I_3 = -25$

Loop 2: $3(I_3 - I_2) + 5I_3 = 15 \Longrightarrow -3I_2 + 8I_3 = 15$

Augmented matrix is $\left[\begin{array}{rrr|r} 10 & -4 & 0 & 10 \\ -4 & 10 & -3 & -25 \\ 0 & -3 & 8 & 15 \end{array}\right]$

4. $A^{-1} = \begin{bmatrix} 5 & -3 & -2 \\ 1 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix}$.

5. (a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 1/2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 6 \\ 0 & -3 & 3 \\ 0 & 0 & 8 \end{bmatrix}$

(b) $L = \begin{bmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

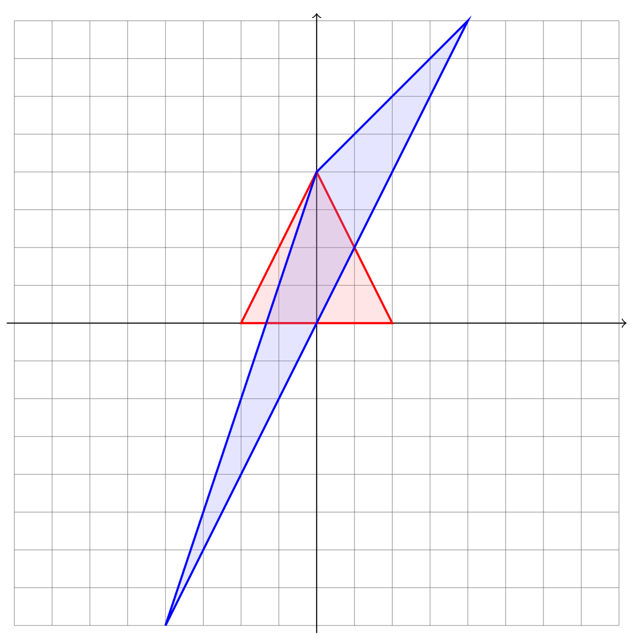
6. (a) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}$

(d) The isosceles triangle.

(e) The scalene triangle.



(f) Area of $(S \circ T)(\mathcal{R}) = \left|\det \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}\right| (\text{Area of } \mathcal{R}) = 16$

7. (a) $T(\mathcal{P}) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \text{Span}\left(\left\{\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right\}\right) = \text{Span}\left(\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right\}\right).$
$T(\mathcal{P})$ is a line since $T(\mathcal{P})$ is a one dimensional subspace.

(b) T is not one-to-one since T transforms both $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

8. (a) $54/5$
(b) 0
(c) $2/135$

9. (a) 6
(b) 15

10. (a) $MN = \begin{bmatrix} 0 & A \\ 0 & I \end{bmatrix}$

(b) Note that MN has a column of zeros, and thus $\det(MN) = 0$. Also, N is a unit lower triangular matrix, and thus $\det N = 1$. Therefore

$$\det M = \det M \det N = \det(MN) = 0.$$

Since $\det M = 0$ we have that M is non-invertible.

11. (a) -3
(b) $\sqrt{17}$

12. $(22/3, -2/3, 17/3)$

13. $h = 10$, $k = 2$

14. (a) $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$
(b) $(-3, 2, -1)$
(c) $\frac{3\sqrt{2}}{2}$

15. Let $H = \left\{\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} : x^2 + y^2 = z^2\right\}$.

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

 Question 15 continues on the next page.

(b) Yes. (Suppose that $A = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \in H$ and $c \in \mathbb{R}$. Then $cA = \begin{bmatrix} cx & cy \\ 0 & cz \end{bmatrix}$, and

$$\begin{aligned}(cx)^2 + (cy)^2 &= c^2x^2 + c^2y^2 \\ &= c^2(x^2 + y^2) \\ &= c^2(z^2) \quad \text{(since } A \in H\text{)} \\ &= (cz)^2.\end{aligned}$$

This shows that $cA \in H$, therefore H is closed under scalar multiplication.)

(c) No. (Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$. Then A and B are in H, but $A + B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \notin H$.)

16. A basis is $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

17. (a) cannot

(b) might

(c) must

(d) must

(e) might, must

18. $\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$