1. (9 points) Evaluate the following limits. Use $-\infty$, ∞ or "does not exist", wherever appropriate.

(a) $\lim_{x\to 4} \dfrac{3\sqrt{x}}{2x-5}$

(b) $\lim_{x\to -3} \dfrac{x+3}{\sqrt{6-x}-3}$

(c) $\lim_{x\to -3^-} \dfrac{4-x}{9-x^2}$

(d) $\lim_{x\to -3} \dfrac{3x^2-|7x-6|}{2x^2+5x-3}$

(e) $\lim_{x\to 1^+} \left(4+\ln(x)\sin\left(\dfrac{2}{x-1}\right)\right)$

2. (5 points) Find the values of a and b that make f continuous everywhere.

$$f(x)=\begin{cases} \dfrac{7+6x-x^2}{x+1} & \text{if } x<-1 \\ ax+b & \text{if } -1\leqslant x\leqslant 4 \\ 2^{5-x}+16 & \text{if } x>4 \end{cases}$$

3. (4 points) Find $f'(x)$ **using the limit definition of the derivative**, where $f(x)=\dfrac{1}{\sqrt{2x-1}}$.

4. (15 points) Find dy/dx for each of the following. Do not simplify your answers.

(a) $y=6^x-2\sqrt[3]{x^5}+\csc(2x+1)+\dfrac{8}{x}$

(b) $y=\dfrac{x^2+e^{2x}}{x+\sqrt{\tan(x^4)}}$

(c) $y=(\ln x)^{\sec x}$

(d) $y=\ln\left(\sqrt[4]{\dfrac{(x+1)^3}{(2x-1)\sin x}}\right)$

(e) $\cos y=\ln(xy)$

5. (4 points) Write an equation of the tangent line to the curve $y^2-4xy=12$ at the point $(-1,2)$.

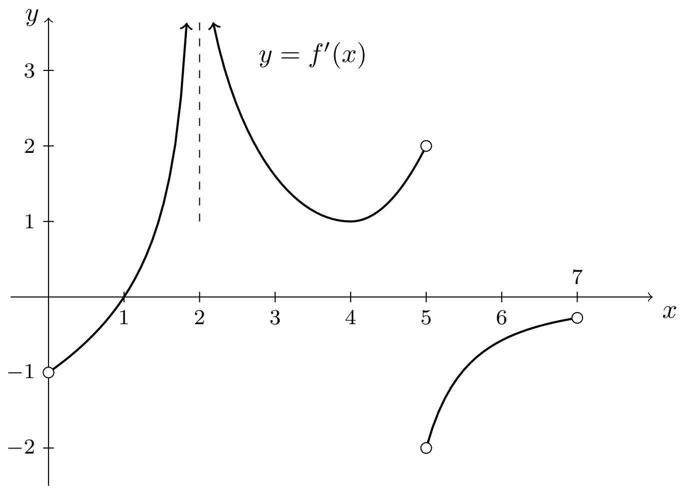
6. (3 points) A particle moves with the position function $s(t)=\dfrac{(t+2)^3}{t^2+1}$, $t\geq 0$.

When does the particle have positive velocity?

7. (5 points) A funnel with the shape of an inverted right circular cone has height 20 cm and radius 5 cm at its top. Water drains out of the bottom of the funnel at a rate of 4 cm^3/sec. At what rate is the height of the water in the funnel decreasing at the moment when the water is 10 cm deep? Recall that the volume of a cone is given by $V=\frac{\pi}{3}r^2h$.

8. (4 points) Find the absolute maximum and minimum values of the function $g(x)=\ln(x^2+x+1)$ on the interval $-1\leqslant x\leqslant 1$.

9. (4 points) Consider the *continuous* function f on $[0,7]$ whose *derivative* f' is given by the following graph.



(a) Write the intervals of increase/decrease of f.

(b) Write the intervals for which f is concave up and the intervals for which f is concave down.

(c) Given that $f(0)=-2$, sketch a graph of f.

10. (12 points) Consider the following function, along with its two first derivatives.

$$f(x)=\frac{x-4}{\sqrt{x^2+8}}\ ,\ f'(x)=\frac{4(x+2)}{\sqrt{(x^2+8)^3}}\ ,$$

$$f''(x)=\frac{-8(x+4)(x-1)}{\sqrt{(x^2+8)^5}}.$$

(a) Find the domain and intercepts of f.

(b) Find the vertical and horizontal asymptotes of f (if any).

(c) Find the intervals of increase/decrease of f.

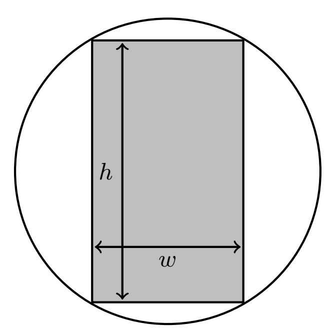
(d) Find the local (relative) extrema of f.

(e) Find the intervals of concavity of f.

(f) Find all points of inflection.

(g) On the next page, sketch a graph of f.

11. (5 points) The strength S of a rectangular beam is proportional to the product of its width w and the square of its height h, thus $S=kwh^2$ where k is a constant. Find the dimensions of the strongest rectangular beam that can be cut from a circular cylindrical log (with bark removed) of radius 10 cm.



12. (3 points) Determine the function $f(x)$ that satisfies $f''(x)=\pi\sin(x)+1$, $f'(\pi)=0$, and $f(0)=\pi$.

13. (4 points) Evaluate $\int_0^3 (8x - 2x^3)\, dx$ by expressing it as a limit of Riemann sums.

You might use the formulas $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ and $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

14. (15 points) Compute the integrals below.

(a) $\int \left(4\sqrt[5]{x^3} + 5e^x + \sqrt{\pi}\right) dx$

(b) $\int \frac{(\sqrt{x}+3)^2}{2x} dx$

(c) $\int \sin x \sec^2 x\, dx$

(d) $\int_0^3 \left|x^2 - 1\right|\, dx$

(e) $\int_0^4 \left(|x-2| - \sqrt{16-x^2}\right) dx$ (interpret as area)

15. (2 points) Suppose f is a continuous function such that

$$\int_{-1}^{1} f(x)\, dx = 3,\ \int_{2}^{3} f(x)\, dx = -2 \text{ and } \int_{1}^{3} f(x)\, dx = 5.$$

Find $\int_{-1}^{2} f(x)\, dx$.

16. Given $g(x) = \int_1^x f(t)\, dt$ and $f(x) = \int_0^{x^2} \sqrt{y+9}\, dy$, find

(a) (1 point) $g(1)$

(b) (3 points) $g''(4)$

17. (2 points) Suppose that $f''(x) > 0$ on the interval (a, b). Prove that the graph of $y = e^{f(x)}$ is concave upwards on (a, b).

Answers

1. (a) 2 (b) -6 (c) $-\infty$ (d) $\frac{11}{7}$ (e) 4 (squeeze theorem)

2. $a = 2$, $b = 10$

3. $f'(x) = \frac{-1}{(2x-1)^{3/2}}$

4. (a) $6^x \ln 6 - \frac{10}{3}x^{2/3} - \csc(2x+1)\cot(2x+1)\cdot 2 - \frac{8}{x^2}$

(b) $\frac{(2x+2e^{2x})\left(x+\sqrt{\tan(x^4)}\right) - (x^2+e^{2x})\left(1+\frac{1}{2}\left(\tan(x^4)\right)^{-1/2}\cdot\sec^2(x^4)\cdot 4x^3\right)}{\left(x+\sqrt{\tan(x^4)}\right)^2}$

(c) $(\ln x)^{\sec x}\left(\sec(x)\tan(x)\ln(\ln x) + \frac{\sec x}{x\ln x}\right)$

(d) $\frac{1}{4}\left(\frac{3}{x+1} - \frac{2}{2x-1} - \frac{\cos x}{\sin x}\right)$

(e) $\frac{-1}{x\left(\sin y + \frac{1}{y}\right)} = \frac{-y}{xy\sin y + x}$

5. $y = x + 3$

6. $[0, 1) \cup (3, +\infty)$

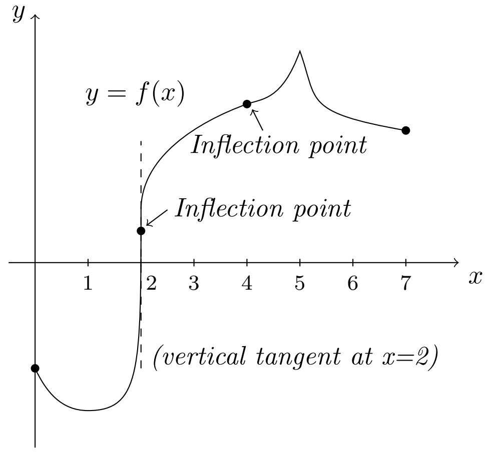
7. $dh/dt = -\frac{16}{25\pi}$ cm/s, so the height is decreasing at the rate of $\frac{16}{25\pi}$ cm/s

8. Abs. max.: $\ln 3$ at $x = 1$. Abs. min.: $\ln\left(\frac{3}{4}\right) = -\ln\left(\frac{4}{3}\right)$ at $x = -\frac{1}{2}$

9. (a) Increasing on $(1, 2) \cup (2, 5)$. Decreasing on $(0, 1) \cup (5, 7)$.

(b) CU on $(0, 2) \cup (4, 5) \cup (5, 7)$. CD on $(2, 4)$.

9. (c)



10. (a) Domain: $\mathbb{R}$. x-int.: $x = 4$. y-int.: $y = -\sqrt{2}$

(b) VA: None. HA: $y = 1$ at $x \to \infty$, $y = -1$ at $x \to -\infty$

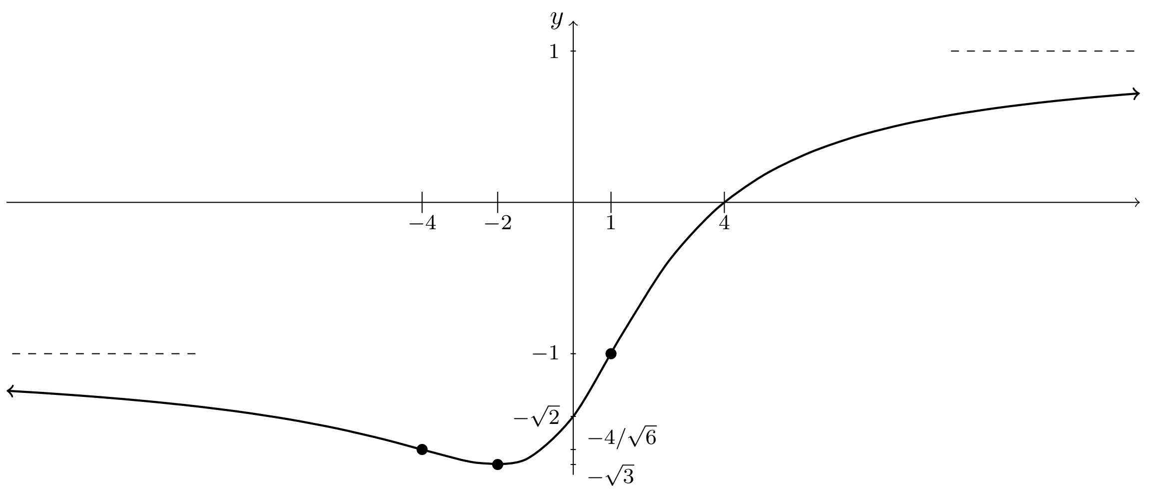
(c) f is increasing on $x > -2$, decreasing on $x < -2$.

(d) Relative min. at $x = -2$, $y = -\sqrt{3}$

(e) f is concave up on $(-4, 1)$,
f is concave down on $(-\infty, -4) \cup (1, \infty)$

(f) Inflection points at $x = -4$, $y = \frac{-4}{\sqrt{6}}$ and $x = 1$, $y = -1$

10. (g)



11. $w = 20/\sqrt{3}$, $h = 20\sqrt{2}/\sqrt{3}$

12. $f(x) = -\pi \sin x + \dfrac{x^2}{2} - 2\pi x + \pi$

13. $\displaystyle\int_0^3 \left(8x - 2x^3\right)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{24i}{n} - \frac{54i^3}{n^3}\right)\frac{3}{n} = -\frac{9}{2}$

14. (a) $\dfrac{5}{2}x^{8/5} + 5e^x + \sqrt{\pi}\cdot x + C$ (b) $\dfrac{1}{2}x + 6x^{1/2} + \dfrac{9}{2}\ln|x| + C$ (c) $\sec x + C$ (d) $\dfrac{22}{3}$ (e) $4 - 4\pi$

15. 10

16. (a) 0 (b) 40

17. If $y = e^{f(x)}$, then $y' = e^{f(x)} \cdot f'(x)$ and $y'' = \left(e^{f(x)} \cdot f'(x)\right) \cdot f'(x) + e^{f(x)} \cdot f''(x) = e^{f(x)} \cdot (f'(x))^2 + e^{f(x)} \cdot f''(x)$.

Factoring, we get that $y'' = e^{f(x)}\left((f'(x))^2 + f''(x)\right)$

Since $f''(x) > 0$ on (a, b), and also $e^{f(x)} > 0$ and $(f'(x))^2 > 0$,
we have that $y'' > 0$ on (a, b),showing that $y = e^{f(x)}$ is concave upwards on (a, b).