1. (8 points) Solve each of the following systems, and give the general solution or indicate that it is inconsistent, as appropriate.

   (a) $\begin{cases} 2x_1 + 4x_2 - 3x_3 - 2x_4 = -7 \\ x_1 + 2x_2 - 2x_3 - x_4 = -5 \\ -2x_1 - 4x_2 + x_3 + x_4 = 3 \end{cases}$

   (b) $\begin{cases} 4x + 8y + 12z = 4 \\ x + y - z = 0 \\ -2x - 5y - 4z = -9 \end{cases}$

2. (5 points) For the system $\begin{cases} x + y + 7z = -7 \\ x + 2y + 10z = -9 \\ y + (k^2 - 6)z = 3k + 7 \end{cases}$

   Find the value(s) of k, if any, for which the system has

   (a) a unique solution

   (b) infinitely many solutions

   (c) no solution

3. (3 points) Given $A = \begin{bmatrix} 2 & 1 & 6 \\ 3 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 2 & 0 \\ 5 & -1 & 1 \end{bmatrix}$.

   Find the following, or indicate that it is not possible, as appropriate:

   (a) AB^T

   (b) $B^T A$

   (c) $(BA^T)^T$

4. (2 points) Are the planes $P_1 : 2x - 3y - z = 6$ and $P_2 : 9x + 5y + 3z = 0$ parallel, perpendicular, or neither?

5. (2 points) Find the value(s) of x and y (if any) so that the matrix $\begin{bmatrix} 2 & 9 & 1 \\ 3x^2 - 3 & x + y & 3 \\ 2y + 5 & 3 & 4 \end{bmatrix}$ is symmetric.

6. (3 points) Given that $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ has $\det(A) = 7$, find the determinant of: $B = \begin{bmatrix} 2a & 2b & 2c \\ g + 2d & h + 2e & i + 2f \\ 3d - 5a & 3e - 5b & 3f - 5c \end{bmatrix}$

7. (4 points) Atreyu is trying to save up to buy his luck dragon, Falkor a new, comfortable harness. As such, he has decided to sell different kinds of lemonade outside his house. To make a bottle of strawberry lemonade, he needs 9 lemons and 10 strawberries. To make a bottle of regular lemonade, he needs 1 lemon and 1 strawberry. To make a bottle of his famous super potent lemonade, he needs 23 lemons and 26 strawberries. If he has 330 lemons and 372 strawberries on hand, how many bottles of each type of lemonade can he make?

   (a) Define your variables and set up the system of equations needed to determine the solution.

   (b) Given that the row reduced form of the associated matrix is $\left[\begin{array}{ccc|c} 1 & 0 & 3 & 42 \\ 0 & 1 & -4 & -48 \end{array}\right]$, write the general solution.

(c) Find all realistic solutions to this problem, given the context of this problem.

8. (5 points) Given the system of equations $\begin{cases} 2x_1 - 2x_2 + x_3 + 4x_4 = -2 \\ 2x_1 + 5x_2 - 3x_3 + x_4 = 0 \\ x_1 + 6x_2 - 7x_3 - 4x_4 = 1 \\ 5x_2 + 2x_3 + x_4 = 0 \end{cases}$

and knowing that $\begin{vmatrix} 2 & -2 & 1 & 4 \\ 2 & 5 & -3 & 1 \\ 1 & 6 & -7 & -4 \\ 0 & 5 & 2 & 1 \end{vmatrix} = 50$, use Cramer's Rule to solve for x_4 ***only***.

9. (5 points) Let A, B and C be 5×5 matrices such that $|A| = 3$, $|B| = -4$ and C is not invertible. Find each determinant, or state that there is not enough information to find it. Justify your answers and show your work.

   (a) $\det(2AB^{-1})$

   (b) $\det(C)$

   (c) $\det(A^T BA^{-1})$

   (d) $\det(CA + CB)$

   (e) $\det(AC + CB)$

10. (5 points) Consider the matrix $A = \begin{bmatrix} 3 & 5 & -7 \\ -1 & 4 & -3 \\ -2 & -1 & 2 \end{bmatrix}$

   (a) Find the adjoint of A.

   (b) Find the determinant of A.

   (c) Use the answers from **a)** and **b)** to find A^{-1}.

11. (5 points) Given points $A(1, 0, 4), B(2, -3, 2)$, and $C(-7, -2, 0)$

   (a) Find the magnitude of $\overrightarrow{AB}$.

   (b) Find a unit vector in the direction of $\overrightarrow{AB}$.

   (c) Write a vector equation for the line through the point C that is parallel to $\overrightarrow{AB}$.

   (d) Find the value of k such that the vector $\mathbf{v} = \langle 1, k, 4 \rangle$ is orthogonal to $\overrightarrow{CA}$.

12. (3 points) Complete the following sentences with the word **MUST**, **MIGHT**, or **CANNOT**, as appropriate:

   (a) If A is an $n \times n$ matrix and $\det(A) \neq 0$, then the system $A\mathbf{x} = \mathbf{0}$ ______________ have infinite solutions.

   (b) If $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ are all non-parallel vectors from $\mathbb{R}^2$, then $\mathbf{w}$ ______________ be a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

   (c) If the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ contains the zero vector, $\mathbf{0}$, then it ______________ be linearly independent.

13. (4 points) Consider the points $P(2, 1, 6), Q(4, 0, 4)$, and $R(0, 1, 9)$

   (a) Write a general equation (of the form $ax + by + cz = d$) for the plane through P, Q, and R.

   (b) Is this plane a subspace of $\mathbb{R}^3$? No justification required.

14. (4 points) Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{array}{rcrcrcl} 3x & - & y & + & 2z & = & 0 \\ \& & & y & + & 4z & = & 0 \end{array} \right\}$.

Is S a subspace? If so, express S as a span of vectors. If not, provide a counterexample.

15. (8 points) Consider the set of vectors $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix}$, $\mathbf{v} \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} -8 \\ 13 \\ 2 \end{bmatrix}$.

(a) Is the vector $(0, 5, 6)$ in span$\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$?

(b) Write the zero vector $\mathbf{0}$ as a **nontrivial** linear combination of the vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$, if possible.

(c) Is the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ linearly independent or linearly dependent?

(d) Describe span$\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$: Is it a point in $\mathbb{R}^3$, a line in $\mathbb{R}^3$, a plane in $\mathbb{R}^3$, or all of $\mathbb{R}^3$?

(e) Provide a basis for span$\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

(f) What is the dimension of span$\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$?

16. (5 points) Let A be a 6×11 matrix whose reduced form contains 4 pivots.

(a) Give the nullity of A.

(b) Circle any possible type of solution for $A\mathbf{x} = \mathbf{b}$

1 **solution** $\quad\infty$ **solutions** $\quad$ **no solution**

(c) Give rank(A^T).

(d) Give the nullity of A^T?

(e) If the matrix product AC has the same rank as A and has a nullity of 0, what is the size $(m \times n)$ of the matrix C?

17. (9 points) Suppose $A = \begin{bmatrix} 4 & -3 & 9 & -7 & 0 \\ 2 & 6 & 5 & -5 & 0 \\ 3 & -4 & 2 & 9 & 0 \\ 8 & -6 & 18 & -14 & 0 \end{bmatrix}$ reduces to $R = \begin{bmatrix} 1 & 0 & 0 & 5 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) Is the set of all columns of A a linearly independent set?

(b) Find a basis for Col(A).

(c) Is Col(A) all of $\mathbb{R}^4$? Briefly justify your answer.

(d) Find a basis for Nul(A).

(e) Is the set of vectors $\{\mathbf{a_1}, \mathbf{a_3}, \mathbf{a_4}\}$ linearly dependent or independent?

(f) Suppose that $\mathbf{x_0} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$ is a particular solution to a system of linear equations $AX = B$. Give the general solution to the system $A\mathbf{x} = \mathbf{b}$ (in vector form).

(g) How many vectors are in Col(A)?

18. (2 points) For what value(s) of a, if any, is the vector $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ in the null space of the matrix $\begin{bmatrix} 6 & 5 & a \\ 3 & 1 & 5 \end{bmatrix}$?

19. (4 points) An economy has two industries: rocks and rolls.
To produce \$1 of rocks requires \$0.40 of rocks and \$0.30 of rolls. To produce \$1 of rolls requires \$0.50 of rocks and \$0.70 of rolls.

(a) How much should each industry produce to satisfy an external demand of \$2100 of rocks and \$1500 of rolls?

(b) Which industries are profitable? Justify your answer in each case.

(c) Give the internal consumption of rocks and rolls when the external demand is as described in a).

20. (4 points) Consider the value of $z = -25x_1 + 10x_2 - 2x_3$,

subject to the constraints $\left\{ \begin{array}{rcrcrcl} 6x_1 & + & 3x_2 & - & 3x_3 & \leq & 6 \\ 3x_1 & + & x_2 & + & 4x_3 & \leq & 7 \\ x_1 \geq 0, & & x_2 \geq 0, & & x_3 & \geq & 0 \end{array} \right\}$.

Use the Simplex Method to find the maximum value of z subject to the constraints (and provide the basic feasible solution), or indicate that the maximum is unbounded, as necessary.

21. (5 points) It has been established that on any given evening, if Tommy Texter is using his cell phone at some moment, there is a 50% chance that he will still be using his cell phone **five** minutes from then. If Tommy is not using his cell phone at some moment, there is a 75% chance that he will be using his cell phone **five** minutes from then.

(a) What is the transition matrix P that describes this situation?

(b) It's 7:00 PM, and Tommy is texting a friend on his cell phone. What is the chance that he will still be using his cell phone at 7:10?

(c) Find a steady-state vector $\mathbf{q}$.

(d) What is the probability that, at the end of a 5-minute interval in the very distant future, Tommy will be using his cell phone?

22. (5 points) The last words of the quotation below were encrypted using the matrix: $A = \begin{bmatrix} 7 & 3 \\ 4 & 3 \end{bmatrix}$

(a) Find the decrytption matrix A^{-1}.

(b) Decode the ciphertext below to complete this line attributed to Mark Twain:
The more I learn about people, the more I like...

JWUIRW

The following table may come in handy:

a	1	3	5	7	9	11	15	17	19	21	23	25
a^{-1}	1	9	21	15	3	19	7	23	11	5	17	25

ANSWERS

1. (a) $\{x_1 = -1 - 2t, x_2 = t, x_3 = 3, x_4 = -2\}$ (b) $\{x = -6, y = 5, z = -1\}$
2. (a) $k \neq \pm 3$ (b) $k = -3$ (c) $k = 3$
3. (a) $\begin{bmatrix} 2 & -6 & 15 \\ 3 & -12 & 17 \end{bmatrix}$ (b) undefined (c) $\begin{bmatrix} 2 & -6 & 15 \\ 3 & -12 & 17 \end{bmatrix}$
4. perpendicular 5. $x = \pm 2, y = -2$ 6. -42
7. (a) Letting x_1 be number of bottles of strawberry lemonade, x_2 be number of bottles of regular lemonade, and x_3 be number of bottles of super potent lemonade, yields $\left\{\begin{array}{rcrcrcr} 9x_1 & + & x_2 & + & 23x_3 & = & 330 \\ 10x_1 & + & x_2 & + & 26x_3 & = & 372 \end{array}\right\}$ (b) $\{x_1 = 42 - 3t, x_2 = -48 + 4t, x_3 = t\}$ (c) $(6, 0, 12)$, $(3, 4, 13)$, $(0, 8, 14)$
8. $\frac{7}{5}$ 9. (a) -24 (b) 0 (c) -4 (d) 0 (e) not enough information
10. $\text{adj}(A) = \begin{bmatrix} 5 & -3 & 13 \\ 8 & -8 & 16 \\ 9 & -7 & 17 \end{bmatrix}$ (b) -8 (c) $A^{-1} = \begin{bmatrix} -5/8 & 3/8 & -13/8 \\ -1 & 1 & -2 \\ -9/8 & 7/8 & -17/8 \end{bmatrix}$
11. (a) $\sqrt{14}$ (b) $\left\langle \frac{1}{\sqrt{14}}, \frac{-3}{\sqrt{14}}, \frac{-2}{\sqrt{14}} \right\rangle$ (c) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -7 \\ -2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}$ (d) $k = -12$
12. (a) CANNOT (b) MUST (c) CANNOT 13. (a) $3x + 2y + 2z = 20$ (b) No.
14. $S = \text{Span}\left\{ \begin{bmatrix} -2 \\ -4 \\ 1 \end{bmatrix} \right\}$ is a subspace
15. (a) Yes. (b) $-6\mathbf{u} + 5\mathbf{v} + \mathbf{w} = \mathbf{0}$ (c) dependent (d) a plane in $\mathbb{R}^3$
(e) $\left\{ \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix} \right\}$ (f) 2
16. (a) 7 (b) infinitely many solutions, no solution (c) 4 (d) 2 (e) 11×4
17. (a) No (b) $\left\{ \begin{bmatrix} 4 \\ 2 \\ 3 \\ 8 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \\ -4 \\ -6 \end{bmatrix}, \begin{bmatrix} 9 \\ 5 \\ 2 \\ 18 \end{bmatrix} \right\}$ (c) No (d) dependent (e) $\left\{ \begin{bmatrix} -5 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$
(f) $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} + s \begin{bmatrix} -5 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ (g) infinitely many 18. $a = 16$
19. (a) \$46000 of rocks; \$51000 of rolls (b) Rocks are profitable; rolls are not.
(c) \$43900 of rocks; \$49500 of rolls
20. Max $x = 28$ at $(0, 3, 1, 0, 0)$
21. (a) 62.5% (b) $\mathbf{q} = \begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix}$ (c) $\frac{3}{5}$ 22. MY DOG