1. (10 marks) Evaluate the following limits. If a limit does not exist, state in which way (∞ or $-\infty$ or *does not exist*).

(a) $\displaystyle\lim_{x\to-3} \frac{2x^2+5x-3}{x^3-9x}$

(b) $\displaystyle\lim_{x\to3} \frac{\sqrt{1+x}-2}{3-x}$

(c) $\displaystyle\lim_{x\to0} \frac{x+1}{\cos(x)-1}$

(d) $\displaystyle\lim_{x\to\infty} \sqrt{\frac{1-8x+9x^3}{x^3+x^2+7}}$

(e) $\displaystyle\lim_{x\to0} \frac{\sin(6x)\sin(x+\pi/6)}{x}$

2. (5 marks) Given

$$f(x) = \begin{cases} k(5x-k) & \text{if } x<1 \\ 2k & \text{if } x=1 \\ x^2+3x+2 & \text{if } x>1 \end{cases}$$

(a) find the value(s) of k, if any, for which $\displaystyle\lim_{x\to1} f(x)$ exists;

(b) find the value(s) of k, if any, for which f is everywhere continuous. Use the definition of continuity to justify your answer.

3. (4 marks) Find the vertical and horizontal asymptotes of $f(x) = \dfrac{2e^x-5}{e^x-1}$

4. Let $f(x) = \dfrac{1}{4x^2}$

(a) (4 marks) Use the limit definition of the derivative to show that the derivative of f is $f'(x) = -\dfrac{1}{2x^3}$

(b) (2 marks) There is one point on the graph of f at which the tangent line is parallel to the line $y = 1-4x$. Find the equation of the tangent line at this point.

5. (16 marks) Find the derivative dy/dx for each of the following. Do not simplify your answers, unless specified otherwise.

(a) $y = \sqrt{x + e^{x\tan x}}$

(b) $y = \dfrac{\sec(4x)-2}{\sqrt{x}+1}$

(c) $y = (\ln x)^{x\ln x}$ (Simplify your answer)

(d) $y = \ln\left(\dfrac{(2x+3)^4}{e^{\sin x}\sqrt{x^6-1}}\right)$

6. (3 marks) Find the second derivative of $f(t) = te^{at}$, where a is a constant. Factor completely.

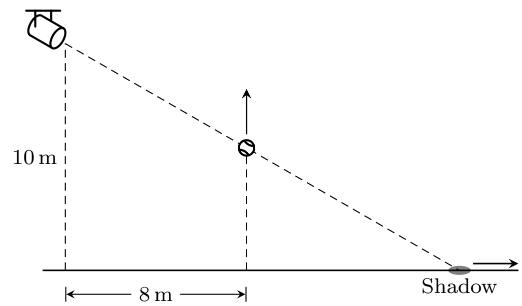
7. (3 marks) Suppose that $h(x) = f\big(xg(x)\big)$, $g(2) = -3$, $f'(-6) = 4$, and $g'(2) = 5$. Find $h'(2)$.

8. (4 marks) Find all points on the curve $x^2y + xy^2 = 16$ where the tangent line is horizontal.

9. (5 marks) Find the absolute maximum and absolute minimum values of

$$f(x) = \frac{4x^2-3}{x^3} \text{ on the interval } \left[\frac{1}{2}, 4\right].$$

10. (6 marks) A bright spotlight illuminates a field from a height of 10 m. Eight metres away from the point on the ground below the spotlight, Jack throws a tennis ball vertically into the air and observes how the ball's shadow moves. When the ball reaches a height of 6 m, its speed is 7 m/s. At that instant, at what speed does the ball's shadow move on the ground?



11. (10 marks) Let $f(x) = \sqrt[3]{2(x-1)(x+2)^2}$. Find:

(a) the domain of f and its x- and y-intercepts,

(b) intervals on which f is increasing/decreasing and local extrema,

(c) intervals on which f is concave up/down and inflection points.

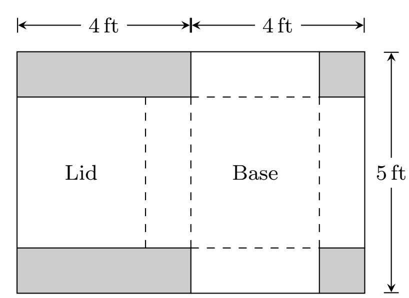
Using this information,

(d) sketch the graph of f on the next page (a grid with axes is given)

If you run out of space on this page, please continue the solution on the next page. The first two derivatives of f are

$$f'(x) = \frac{2x}{\sqrt[3]{4(x-1)^2(x+2)}} \qquad f''(x) = \frac{-4}{\sqrt[3]{4(x-1)^5(x+2)^4}}$$

12. (6 marks) A box with a lid is to be constructed from a 5 ft by 8 ft rectangular piece of cardboard by cutting out squares from two of the corners and strips from the other corners, bending up the sides, and folding the lid across the top to cover the base. Find the dimensions of the box having the largest possible volume.



13. (4 marks) Find a function f such that $f''(x) = x - \cos x$, given that the line $y = 2x-2$ is tangent to its graph at $x = 0$.

14. (3 marks) Consider the limit $\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2i^5}{n^6}-\frac{i^4}{n^5}\right)$

Express this limit as a definite integral over the interval $[0,1]$ by interpreting the sum as a Riemann sum.

Do not evaluate the limit or the integral.

15. (8 marks) Evaluate each of the following integrals.

(a) $\int\left(\frac{4}{\sqrt[4]{x^7}}+\frac{e^x}{2}+e^{\sqrt{\pi}}\right)dx$

(b) $\int_1^4 \frac{\left(2\sqrt{x}-x\right)^2}{x^3}\,dx$

(c) $\int 2\left(3\sec x-5x\cos x\right)\sec x\,dx$

(d) $\int_1^e \frac{d}{dx}\left[\frac{x\ln x}{1+x^2}\right]dx$

16. (4 marks) Let $A(x)=\int_{1/x}^{2}\frac{t}{t^2+1}\,dt$.

(a) What is $A\left(\frac{1}{2}\right)$?

(b) Find $A'(x)$ and simplify.

17. Suppose that f is such that $f''(x)$ exists and is continuous for all x. Suppose further that

$$f(0)=2,\qquad f(1)=5\qquad\text{and}\qquad f(2)=8$$

.

(a) (2 marks) Use the Mean Value Theorem to show that $f'(x)=3$ at two different points.

(b) (1 mark) Use part (a) and Rolle's theorem to show that there is a point such that $f''(x)=0$.

Answers

1. (a) $-\frac{7}{18}$ (b) $-\frac{1}{4}$ (c) $-\infty$ (d) 3 (e) 3

2. (a) $k=2$ or $k=3$ (b) $k=3$

3. VA: $x=0$. HA: $y=2$ and $y=5$

4. (b) $y=-4\left(x-\frac{1}{2}\right)+1$

5. (a) $y'=\frac{1}{2}\left(x+e^{x\tan x}\right)^{-1/2}\left(1+e^{x\tan x}\left(1\cdot\tan x+x\cdot\sec^2 x\right)\right)$

(b) $y'=\dfrac{\sec(4x)\tan(4x)\cdot 4\cdot(\sqrt{x}+1)-(\sec(4x)-2)\cdot\frac{1}{2\sqrt{x}}}{\left(\sqrt{x}+1\right)^2}$

(c) $y'=(\ln x)^{x\ln x}\left(\ln(\ln x)\cdot(1+\ln x)+1\right)$

(d) $y'=\frac{8}{2x+3}-\cos x-\frac{3x^5}{x^6-1}$

6. $f''(t)=ae^{at}(2+at)$

7. 28

8. (2,-4)

9. Absolute minimum value of $y=-16$ at $x=\frac{1}{2}$. Absolute maximum value of $y=\frac{16}{9}$ at $x=\frac{3}{2}$

10. 35 m/s

11. (a) Domain: $\mathbb{R}$. x-int.: $x=1$, $x=-2$. y-int.: $y=-2$

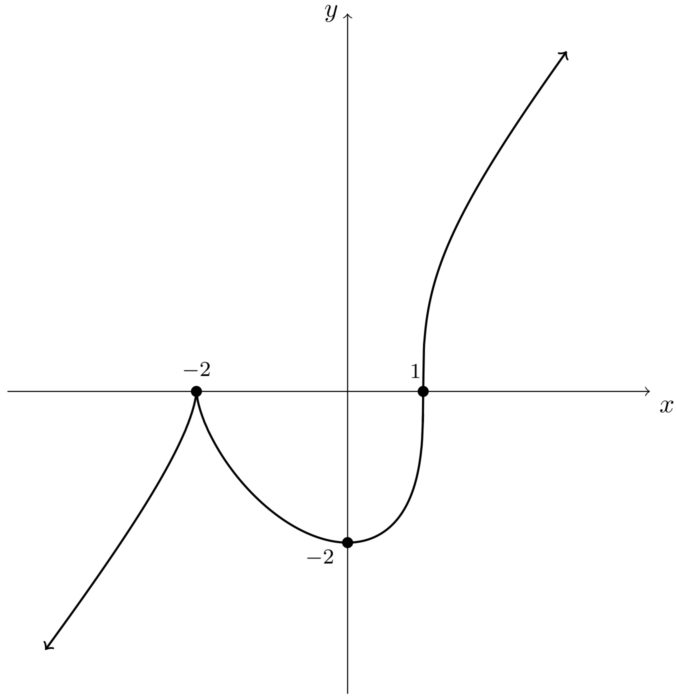
(b) f is increasing on $(-\infty,-2)\cup(0,\infty)$, decreasing on $(-2,0)$

(c) Local max at $x=-2$, $y=0$. Local min at $x=0$, $y=-2$

(d) f is concave up on $(-\infty,-2)\cup(-2,1)$
f is concave down on $(1,\infty)$

(e) Inflection point at $x=1$, $y=0$

(f)



12. 1ft × 3ft × 3ft

13. $f(x) = \frac{x^3}{6} + \cos x + 2x - 3$

14. $\int_0^1 \left(2x^5 - x^4\right)\, dx$

15. (a) $-\frac{16}{3}x^{-3/4} + \frac{1}{2}e^x + e^{\sqrt{\pi}} \cdot x + C$

(b) $\ln(4) - 1$

(c) $6\tan x - 5x^2 + C$

(d) $\frac{e}{1+e^2}$

16. (a) 0

(b) $\frac{1}{x\left(x^2+1\right)}$

17. (a) Since f'' exists, f' exists and thus f is continuous, for all x.
Therefore f is continuous on [0,1] and differentiable on (0,1).
By the Mean Value Theorem, there is a number $c \in (0,1)$ such that $f'(c) = \frac{f(1)-f(0)}{1-0} = 3$
Similarly, f is continuous on [1,2] and differentiable on (1,2).
By the Mean Value Theorem, there is a number $d \in (0,1)$ such that $f'(d) = \frac{f(2)-f(1)}{2-1} = 3$

(b) Since f'' exists, f' is continuous for all x.
Let c and d be the two x-values found in part (a). Note that they are different since $c \in (0,1)$ and $d \in (1,2)$
f' is continuous on $[c,d]$, f' is differentiable on (c,d) (since f'' exists), and $f'(c) = f'(d) = 3$.
By Rolle's theorem, there is a number $k \in (c,d)$ such that $f''(k) = 0$