(8) 1. Solve each of the following systems, or show that it is inconsistent, as appropriate.

a) $\begin{cases} 3x_1 - 21x_2 \qquad + 18x_4 = 15 \\ -2x_1 + 14x_2 - 7x_3 + 2x_4 = 11 \\ -x_1 + 7x_2 - 4x_3 + 2x_4 = 7 \end{cases}$ b) $\begin{cases} 2x - y + 5z = 10 \\ x + y - 3z = -2 \\ 2x + 4y + z = 1 \end{cases}$

(6) 2. For the system $\begin{cases} -x + 2y + z = 3 \\ 4x - 9y - 8z = -5 \\ 8x - 19y + 5hz = 2k + 1 \end{cases}$ find the value(s) of h and k, if any, for which the system has

a) infinitely many solutions b) no solution c) a unique solution

(4) 3. The chef at the Soup Bowl restaurant is well-known for his three famous soups: The Medley, Mighty Green, and Carrotastic. Aside from other ingredients, the main ingredients for these soups are spinach and carrots. To make one batch of The Medley, the chef uses 4 cups of spinach and 2 cups of carrots. To make one batch of the Mighty Green, the chef uses 10 cups of spinach and 2 cups of carrots. To make one batch of the Carrotastic, the chef uses no spinach and 3 cups of carrots. The chef has 70 cups of spinach and 20 cups of carrots. If all ingredients must be used and only complete batches are allowed, what are the possible number of batches of each type of soup that can be made?

a) Define your variables and set up the system of equations needed to determine the solution.

b) Given that the general solution to the system is $\begin{cases} x = -\frac{5}{2}t + 5 \\ y = t + 5 \\ z = t \end{cases}$, find all realistic solutions.

(4) 4. Given $A = \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 3 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 3 & -2 \\ 0 & 6 \\ 5 & 0 \end{bmatrix}$

Find the following, or indicate that it is not possible, as appropriate:

a) $A^{-1} + 3I$ b) BA^2 c) $(2BC)^T$

(3) 5. Find all values of a and b in order for the matrix $\begin{bmatrix} 7 & a+2b & a-b \\ 7 & 2a & 4 \\ 1 & 4 & 2b \end{bmatrix}$ to be symmetric.

(6) 6. Suppose A and B are 4×4 matrices such that $|A| = -\frac{1}{3}$ and $|B| = 2$. Compute the following determinants, if possible.

a) $\det((2A)^{-1})$ b) $\det(3(AB)^T)$ c) $\det(A + B)$ d) $\det\left(B^{-1}\operatorname{adj}(B)\right)$

(4) 7. Given that $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 6$, find

a) $\begin{vmatrix} 2d+3a & 2e+3b & 2f+3c \\ a & b & c \\ g & h & i \end{vmatrix}$ b) $\begin{vmatrix} a & 2a & d \\ b & 2b & e \\ c & 2c & f \end{vmatrix}$

(3) 8. Use Cramer's rule to solve the linear system $AX = B$ for y only. Note that $|A| = 7$.

$$\begin{cases} 3x - 6y - 2z = 4 \\ 2x + y - 7z = 1 \\ -x + 3y \qquad = -3 \end{cases}$$

(6) 9. Given $A = \begin{bmatrix} -2 & 1 & 0 \\ 0 & 4 & -3 \\ -5 & 2 & 0 \end{bmatrix}$

a) Compute $\operatorname{adj}(A)$
b) Find $\det(A)$
c) Use parts a) and b) to find A^{-1}
d) Use A^{-1} to solve the linear system $AX = B$ for $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ -6 \\ 9 \end{bmatrix}$

(3) 10. Consider the vectors $\boldsymbol{u} = (k, 4{,}8)$, $\boldsymbol{v} = (2, -3, -6)$, and $\boldsymbol{w} = (-4{,}0{,}3)$

a) Find the components of the unit vector that points in the direction opposite of $\boldsymbol{w}$.
b) For what value(s) of k will the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ be orthogonal?
c) The vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ form two sides of a triangle. What is the length of that triangle's third side?

(4) 11. Consider the points $P(2{,}6, -1)$, $Q(-3{,}3{,}0)$, and $R(1, -1{,}2)$.

a) Write the parametric equations of the line that passes through the points P and Q.
b) Show that the points P, Q, and R are not collinear.
c) Write the equation of the plane in $\mathbb{R}^3$ containing the points P, Q, and R in standard form $ax + by + cz = d$.

(3) 12. Consider the linear system $\begin{cases} x_1 - 2x_2 + \phantom{x_3 +{}} x_4 = -3 \\ \phantom{x_1 - 2x_2 + {}} x_3 + 5x_4 = 2 \end{cases}$

a) Write the solution to the linear system in vector form.
b) Identify two vectors which are parallel to the plane described in part a).
c) Is the point $(-1{,}1{,}2{,}0)$ on the plane described in part a)?

(3) 13. Suppose A is a 5×5 matrix. Label each statement as true or false. Justify.

a) If the nullity of A is 0, then $\det(A) = 0$.
b) If the dimension of $\operatorname{Nul}(A)$ is 2, then the dimension of $\operatorname{Col}(A)$ is 3.
c) If the dimension of $\operatorname{Col}(A)$ is 4, then the dimension of $\operatorname{Col}(A^T)$ is 1.

(5) 14. Consider the sets $S_1 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \middle| x \leq y \right\}$ and $S_2 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \middle| \begin{matrix} x - 2y = 0 \\ \text{and} \\ y - 3z = 0 \end{matrix} \right\}$

a) Are $\vec{0}$ and $\begin{bmatrix} 2 \\ -5 \end{bmatrix}$ in S_1? Justify.
b) Is S_1 closed under scalar multiplication? Justify.
c) Is S_1 a subspace of $\mathbb{R}^2$?
d) Give an example of a nonzero vector in S_2.
e) Is S_2 a subspace of $\mathbb{R}^3$? If so, express S_2 as a span of vector(s). If not, provide a counterexample.

(6) 15. Consider the vectors $\boldsymbol{u_1} = (1,7,2), \boldsymbol{u_2} = (-1,3,3)$, and $\boldsymbol{u_3} = (6,2,-8)$.

a) Determine whether or not the set $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ is linearly dependent or linearly independent. If the set is linearly dependent, then find a dependency equation.
b) Is the span of $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ a point, a line, a plane, or $\mathbb{R}^3$?
c) Is the vector $\boldsymbol{w} = 2\boldsymbol{u_1} - 3\boldsymbol{u_2}$ in span$\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$?
d) What is the dimension of span$\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$?

(2) 16. Find a basis for the subspace $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \middle| x - 2y + 5z = 0 \right\}$. What is the dimension of S?

(8) 17. Let $\boldsymbol{u_1} = (-1,-1,2,3), \boldsymbol{u_2} = (1,2,-1,5), \boldsymbol{u_3} = (-2,-2,4,6), \boldsymbol{u_4} = (4,-3,-1,5)$, and

$\boldsymbol{u_5} = (8,13,-11,16)$.

If $A = [\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}, \boldsymbol{u_5}] = \begin{bmatrix} -1 & 1 & -2 & 4 & 8 \\ -1 & 2 & -2 & -3 & 13 \\ 2 & -1 & 4 & -1 & -11 \\ 3 & 5 & 6 & 5 & 16 \end{bmatrix}$ reduces to $R = \begin{bmatrix} 1 & 0 & 2 & 0 & -3 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

a) Select a basis for the span of the vectors $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}$, and $\boldsymbol{u_5}$.
b) Write $\boldsymbol{u_5}$ as a linear combination of the basis vectors selected in a).
c) Find a basis for Nul(A).
d) Find the value of k such that the vector $\boldsymbol{b} = (k,-5,3,0,1)$ is in Nul(A).
e) What is the rank of A and the nullity of A?
f) Is the set of vectors $\{\boldsymbol{u_3}, \boldsymbol{u_4}, \boldsymbol{u_5}\}$ linearly dependent or independent? Justify.

(3) 18. Let $A = \begin{bmatrix} -33 & -11 & -55 \\ 27 & 9 & 45 \\ -9 & -3 & -15 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 11 \\ -9 \\ 3 \end{bmatrix}$

a) Given that $\boldsymbol{u_1} = (-\frac{1}{3}, 1, 0)$ and $\boldsymbol{u_2} = (-\frac{5}{3}, 0, 1)$ span Nul(A), write the general solution to $A\boldsymbol{x} = \boldsymbol{0}$.
b) Show that $\boldsymbol{v} = (-6,2,3)$ is a solution to $A\boldsymbol{x} = \boldsymbol{b}$.
c) Write the general solution to $A\boldsymbol{x} = \boldsymbol{b}$.

(5) 19. Suppose we want to analyze an open economy with only two sectors: transportation and food. It costs \$0.40 of transportation and \$0.20 of food to produce \$1 worth of transportation, and it costs \$0.30 of transportation and \$0.70 of food to produce \$1 worth of food.

a) Identify which, if any, sectors in this economy are profitable.
b) Is the economy productive?
c) How many dollars worth of transportation and food should be produced in order to satisfy an outside demand of \$600 of transportation and \$300 of food?

(4) 20. Given the constraints for a linear program:

$$\begin{cases} x - \ y \leq \ 8 \\ -\frac{3}{2}x + \ y \leq 14 \\ -x + 2y \leq 24 \\ x \geq 0, y \geq 0 \end{cases}$$

Use the Simplex Method to **MINIMIZE** $z = 2x - 10y$ subject to those constraints and provide the basic feasible solution (including slack variables). If you find that the minimum is unbounded, support your answer by naming an unbounded variable.

(5) 21. In any given hour John is either hungry or not hungry. In the case he is hungry at any given hour, he will try to eat something, but there is 20% chance that he will also be hungry in the next hour. In the case he is not hungry at a given hour, there is 60% chance that he will be hungry in the next hour.

a) Find the transition matrix P associated with this situation.
b) If at 1pm John is not hungry, what is the chance that he will be hungry at 3pm?
c) Find a steady-state vector $\boldsymbol{q}$.
d) What is the probability that in a very distant future John will be hungry?

(5) 22. In his work Sam encrypts sensitive information using a Hill 2-cipher. Last encoding matrix he and his team used was $A = \begin{bmatrix} 2 & 7 \\ 5 & 3 \end{bmatrix}$, so just for fun he used it to let his team know what is the name of his newborn baby: **KXDEXG**

a) Find the decryption matrix A^{-1}.
b) Decode the name of the baby.

The following tables may be useful:

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26 or 0

a	1	3	5	7	9	11	15	17	19	21	23	25
a^{-1}	1	9	21	15	3	19	7	23	11	5	17	25

ANSWERS

1a) $\{x_1 = 5 + 7s - 6t, x_2 = s, x_3 = -3 + 2t, x_4 = t\}$ b) $\{x = 2, y = -1, z = 1\}$

2a) $h = -4, k = -2$ b) $h = -4, k \neq -2$ c) $h \neq -4$

3a) $\begin{cases} 4x + 10y = 70 \\ 2x + 2y + 3z = 20 \end{cases}$ $x =$ number of batches of The Medley, $y =$ number of batches of Mighty Green, $z =$ number of batches of Carrotastic b) $\{x = 5, y = 5, z = 0\}$ or $\{x = 0, y = 7, z = 2\}$

4a) $\begin{bmatrix} 3 & -1 \\ 1/3 & 11/3 \end{bmatrix}$ b) not possible c) $\begin{bmatrix} 34 & 10 \\ -40 & 36 \end{bmatrix}$

5) $a = 3, b = 2$

6a) $-3/16$ b) -54 c) cannot be computed d) 4

7a) -12 b) 0

8) $y = -25/7$

9a) $\begin{bmatrix} 6 & 0 & -3 \\ 15 & 0 & -6 \\ 20 & -1 & -8 \end{bmatrix}$ b) 3 c) $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 0 & -2 \\ 20/3 & -1/3 & -8/3 \end{bmatrix}$ d) $X = \begin{bmatrix} -3 \\ -3 \\ -2 \end{bmatrix}$

10a) $(4/5, 0, -3/5)$ b) $k = 30$ c) $3\sqrt{14}$

11a) $\begin{cases} x = -5t + 2 \\ y = -3t + 6 \\ z = t - 1 \end{cases}$ b) $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = t\begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}$ no such t c) $x - 7y - 16z = -24$

12a) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ -5 \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \end{bmatrix}$ b) $\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -5 \\ 1 \end{bmatrix}$ c) yes

13a) False b) True c) False

14a) $\mathbf{0} \in S_1, \begin{bmatrix} 2 \\ -5 \end{bmatrix} \notin S_1$ b) No. Consider $-2\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ c) No d) $\begin{bmatrix} 12 \\ 6 \\ 2 \end{bmatrix}$ e) Yes. $S_2 = \text{span}\left\{\begin{bmatrix} 2 \\ 1 \\ 1/3 \end{bmatrix}\right\}$

15a) L.D. $-2\boldsymbol{u_1} + 4\boldsymbol{u_2} + \boldsymbol{u_3} = \mathbf{0}$ b) a plane c) yes d) 2

16) $\left\{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}\right\}$ $\text{Dim}(S) = 2$

17a) $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_4}\}$ b) $\boldsymbol{u_5} = -3\boldsymbol{u_1} + 5\boldsymbol{u_2}$ c) $\left\{\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$

d) $k = -3$ e) $\text{rank}(A) = 3, \text{nullity}(A) = 2$ f) L.I.

18a) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5/3 \\ 0 \\ 1 \end{bmatrix}$ b) $\begin{bmatrix} -33 & -11 & -55 \\ 27 & 9 & 45 \\ -9 & -3 & -15 \end{bmatrix}\begin{bmatrix} -6 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 11 \\ -9 \\ 3 \end{bmatrix}$

c) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5/3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -6 \\ 2 \\ 3 \end{bmatrix}$

19a) transportation b) yes c) \$2250 of transportation, \$2500 of food

20) Min $z = -240$ when $x = 40, y = 32, s_1 = 0, s_2 = 42, s_3 = 0$

21a) $\begin{bmatrix} 0.2 & 0.6 \\ 0.8 & 0.4 \end{bmatrix}$ b) 36% c) $\begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix}$ d) $3/7$

22a) $\begin{bmatrix} 25 & 11 \\ 19 & 8 \end{bmatrix}$ b) SKYLAR