1. Use power series to evaluate the limit

$$\lim_{x\to 0} \frac{\sin^2 x}{\ln(1-x)+\sin x}$$

2. (a) Use the binomial series to find the Maclaurin series for $f(x) = \dfrac{1}{\sqrt{1-x^2}}$ and give its radius of convergence.

   (b) Using this series, find the Maclaurin series for $\arccos(x)$ (Hint: $(\arccos(x))' = -\dfrac{1}{\sqrt{1-x^2}}$).

3. Let $g(x) = \displaystyle\int_0^x (\arctan(t^3) + \sin(t^3))\, dt$

   (a) Find the Maclaurin series for $g(x)$; express your answer in $\sum$ form and give its radius of convergence.

   (b) Find $g(0.5)$ correct to 6 decimal places.

4. For the function $f(x) = x^{2/3}$:

   (a) Find the third degree Taylor polynomial $T_3(x)$ centered at $a = 1$ and an expression for the remainder $R_3(x)$.

   (b) Estimate the maximum error of approximation $f(x) \simeq T_3(x)$ when x lies in the interval $[0.5, 1.5]$.

5. Let $\mathcal{C}$ be the curve with parametric equations: $\begin{cases} x = 1-t^2 \\ y = t^3 - 4t \end{cases} \quad t \in \mathbb{R}$

   (a) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

   (b) Locate all intercepts and points of horizontal or vertical tangency.

   (c) Use the above information to help you sketch $\mathcal{C}$; show the orientation of the curve.

   (d) The curve forms a loop. Set up (**do not evaluate**) the integral needed to find the total area inside the loop.

6. (a) Sketch the polar curves $r_1 = \sin\theta$ and $r_2 = \cos(2\theta)$ on the same axes.

   (b) Find all the points of intersection for $\theta \in [0, 2\pi]$.

   (c) Set up (**do not evaluate**) the integral needed to calculate the area inside the first curve (r_1) and outside the second (r_2).

   (d) Set up (**do not evaluate**) the integral needed to calculate the length of one petal of the rose curve $r_2 = \cos(2\theta)$.

7. Let $\mathcal{C}$ be the space curve represented by $\mathbf{r}(t) = \left\langle \dfrac{t^2}{2},\ \ln t,\ t\sqrt{2} \right\rangle$, where $t > 0$.

   (a) Find the length of the curve from $t = 1$ to $t = 5$.

   (b) Find the curvature at $t = 1$.

   (c) Find the tangential and normal components of the acceleration vector (a_T and a_N) at $t = 1$.

8. Let $\mathbf{r}(t)$ be any smooth space curve with unit tangent vector $\mathbf{T}$ and binormal vector $\mathbf{B}$. Show that $\mathbf{B}' \cdot \mathbf{T} = 0$ (Hint: Start with $\mathbf{B} \cdot \mathbf{T} = 0$).

9. Sketch and give the name of the following surfaces:
   (a) $y = x^2 + 4z^2 + 1$
   (b) $z = \sqrt{r^2 - 9}$
   (c) $\rho = 4 \sin\phi \cos\theta$

10. Find the limit if it exists or show that it does not exist.
   (a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4y^3}{x^4 + 2y^4}$
   (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4y^4}{(x^2 + y^4)^3}$

11. Consider the level surface $\mathcal{S}$: $F(x, y, z) = 2x^2 + 2y^2 - z^2 = 0$ and the point $P(1, 1, 2)$.
   (a) Find the direction in which the maximum rate of change of F at P occurs.
   (b) What is the maximum rate of change?
   (c) Find the directional derivative of F at P in the direction $\mathbf{v} = \langle 2, -1, -2 \rangle$.
   (d) Find a vector tangent to the curve of intersection of $\mathcal{S}$ and the hyperbolic paraboloid $z = 3x^2 - y^2$ at P.

12. Let $z = f(x, y) = \sqrt{2x^3 + y^2}$.
   (a) Find the total differential of f.
   (b) Find an equation of the tangent plane to this surface at $P(2, 3, 5)$.
   (c) Use this tangent plane to approximate $f(2.02, 2.97)$.

13. Let $z = f(u)$ where $u = x^2 + y^2$ Show that

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 4(x^2 - y^2)\frac{d^2 f}{du^2}$$

14. Find and classify the critical points of $f(x, y) = x^3 + y^3 - 3x^2 - 3y^2 - 9x$.

15. Use the method of Lagrange multipliers to find the maximum and minimum of $f(x, y, z) = z$ subject to the constraints $g(x, y, z) = x + y + z - 12 = 0$ and $h(x, y, z) = x^2 + y^2 - z = 0$.

16. Set up the double integral in cartesian coordinates (**do not evaluate**): $\displaystyle\int_0^{\pi/2}\int_0^{2\cos\theta} r^3 \sin(2\theta)\, dr d\theta$

17. Evaluate the integrals (sketch the regions).
   (a) $\displaystyle\int_0^1 \int_{\arctan y}^{\pi/4} \sec x \, dxdy$
   (b) $\displaystyle\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+y^2}^{4} (x^2 + y^2) dzdydx$

18. Let $\mathcal{S}$ be the solid region above the cone $z = \sqrt{\dfrac{x^2+y^2}{3}}$, and below the sphere $x^2+y^2+z^2=9$. Evaluate $\displaystyle\iiint_{\mathcal{S}} \sqrt{x^2+y^2+z^2}\,dV$.

19. Sketch the solid region $\mathcal{S}$ in the first octant bounded by the coordinate planes and the surfaces $z = 1-x^2$ and $x+y=1$. Set up (**do not evaluate**) a triple integral needed to find its volume.

Answers

1. -2

2. (a) $\displaystyle\frac{1}{\sqrt{1-x^2}} = 1 + \frac{x^2}{2} + \sum_{n=2}^{\infty} \frac{(2n-1)!!x^{2n}}{2^n n!}$ $\qquad R = 1$

   (b) $\displaystyle\arccos x = \frac{\pi}{2} - \left(x + \frac{x^3}{6} + \sum_{n=2}^{\infty} \frac{(2n-1)!!x^{2n+1}}{2^n n!(2n+1)}\right)$ $\qquad R = 1$

3. (a) $\displaystyle g(x) = \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2n+1} + \frac{1}{(2n+1)!}\right) \frac{x^{6n+4}}{6n+4}$ $\qquad R = 1$

   (b) $g(0.5) \simeq 0.031201$

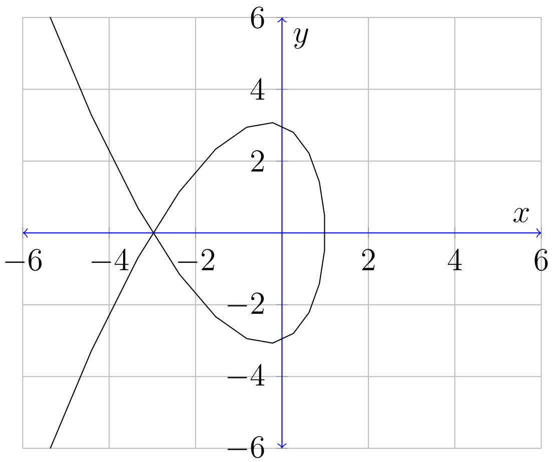
4. (a) $\displaystyle T_3(x) = 1 + \frac{2(x-1)}{3} - \frac{(x-1)^2}{9} + \frac{4(x-1)^3}{81}$

   $\displaystyle R_3(x) = \frac{-7(x-1)^4}{243z^{10/3}}$ where z is between x and 1

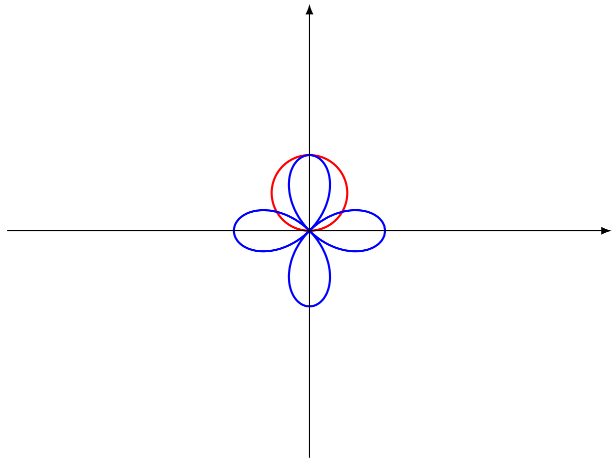
   (b) $|R_3(x)| \le 0.018147$

5. (a) $\displaystyle\frac{dy}{dx} = \frac{4-3t^2}{2t}$ and $\displaystyle\frac{d^2y}{dx^2} = \frac{3t^2+4}{4t^3}$

   (b) Intercepts: $(0, \pm 3)$ $(t = \pm 1)$, $(-3, 0)$ $(t = \pm 2)$ and $(1, 0)(t = 0)$, V.T. at $(1, 0)$ $(t = 0)$ and H.T. at $(-1/3, \pm 16\sqrt{3}/9)$ $(t = \pm 2/\sqrt{3})$

   (c) 

   (d) $\displaystyle A = 2\int_0^2 (t^3-4t)(-2t)dt = -4\int_0^2 (t^4-4t^2)dt$



6. (a)
   (b) Points of intersection are $(0,0)$, $(1/2,\pi/6)$, $(1/2,5\pi/6)$, $(-1,3\pi/2)=(1,\pi/2)$
   (c) $A = 2(1/2)\displaystyle\int_{\pi/6}^{\pi/2} \{\sin^2\theta - \cos^2(2\theta)\}\, d\theta$
   (d) $\mathcal{L} = 2\displaystyle\int_0^{\pi/4} \sqrt{1+3\sin^2(2\theta)}\, d\theta$

7. (a) $\mathcal{L} = 12 + \ln 5$
   (b) $\kappa(1) = \frac{\sqrt{2}}{4}$
   (c) $a_T(1) = 0$ and $a_N(1) = \sqrt{2}$

8. Note that $\mathbf{B}'\cdot\mathbf{T}+\mathbf{B}\cdot\mathbf{T}' = 0$, $\mathbf{T}' = \|\mathbf{T}'\|\mathbf{N}$ and $\mathbf{B}\cdot\mathbf{N} = 0$. It follows that $\mathbf{B}\cdot\mathbf{T}' = \|\mathbf{T}'\|\mathbf{B}\cdot\mathbf{N} = 0$ so $\mathbf{B}'\cdot\mathbf{T} = 0$

9. (a) elliptic paraboloid ($y-1 = x^2+4z^2$)
   (b) Top half of Hyperboloid of one sheet
   (c) Sphere ($(x-2)^2+y^2+z^2 = 4$)

10. (a) The limit is 0.
    (b) The limit does not exist.

11. (a) $\dfrac{\langle 1,1,-1\rangle}{\sqrt{3}}$
    (b) $4\sqrt{3}$
    (c) $D_{\mathbf{u}}f(P) = 4$
    (d) $\langle 3,5,8\rangle$

12. (a) $dz = \dfrac{3x^2}{\sqrt{2x^3+y^2}}dx + \dfrac{y}{\sqrt{2x^3+y^2}}dy$
    (b) $z-5 = \frac{12}{5}(x-2) + \frac{3}{5}(y-3)$
    (c) $f(2.02, 2, 97) \simeq 5.03$

13. $\dfrac{\partial^2 z}{\partial x^2} = 2\dfrac{df}{du} + 4x^2\dfrac{d^2 f}{du^2}$ due to symmetry: $\dfrac{\partial^2 z}{\partial y^2} = 2\dfrac{df}{du} + 4y^2\dfrac{d^2 f}{du^2}$

    Subtract to get the given equation.

14. There are 4 critical points. The points $(-1, 2)$, $(3, 0)$ are all saddle points, $(3, 2)$ is a local minimum and $(-1, 0)$ is a local maximum.

15. The maximum is $f(-3, -3, 18) = 18$ and the minimum is $f(2, 2, 8) = 8$.

16. $\displaystyle\int_0^2 \int_0^{\sqrt{2x-x^2}} 2xy\,dy\,dx$

17. (a) $\sqrt{2} - 1$

    (b) $\dfrac{32\pi}{3}$

18. $\dfrac{81\pi}{4}$

19. $\displaystyle\int_0^1 \int_0^{1-x} \int_0^{1-x^2} dz\,dy\,dx$