

1. (2 points) A clerk at the Acme Gadget Company needs your help to determine feasible production levels for three different gadgets. He defined  $x$  as the number of Widgets,  $y$  as the number of Whatnots, and  $z$  as the number Gizmos and he solved a system of equations that took into account constraints on time and resources. The infinitely many solutions he found are given below, but he is unsure of what to do next. Help him to find all *realistic* solutions to his problem.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 - \frac{2}{3}t \\ \frac{1}{3}t - 4 \\ t \end{bmatrix}$$

2. (2 points) Given  $A^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ ,  $B = \begin{bmatrix} 2 & 1 \\ -2 & 2 \end{bmatrix}$ , find  $(AB)^T$  if defined.

3. Let  $\vec{u} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$  and  $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ .

- (a) (1 point) Are  $\vec{u}$  and  $\vec{v}$  parallel, perpendicular, or neither? Justify.  
 (b) (3 points) Find a vector equation of the line that is orthogonal to  $\text{Span}(\vec{u}, \vec{v})$  and that passes through the point  $P(3, 0, 1)$ .

- (c) (2 points) Let  $\vec{w} = \begin{bmatrix} k \\ 1 \\ 7 \end{bmatrix}$ .

For what values of  $k$  will the set  $\{\vec{u}, \vec{v}, \vec{w}\}$  form a basis for  $\mathbb{R}^3$ ?

4. (4 points) Use a determinant to determine whether the set is linearly independent or linearly dependent.

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

5. (3 points) Suppose  $A$  is a  $4 \times 4$  matrix with  $\det(A^T) = 2$ . Answer TRUE or FALSE for each part below and briefly justify:

- (a) The nullity of  $A^T$  is 2.  
 (b)  $\det(2A^{-1}) = 8$   
 (c)  $A$  is not invertible.

6. (1 point) What is the minimum number of vectors needed to span a plane in  $\mathbb{R}^3$ ?

7. (a) (2 points) Find an equation of the plane (in the form  $ax + by + cz = d$ ) which passes through the point  $(1, 2, 3)$  and which is orthogonal to the vector  $\vec{n} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ .

- (b) (2 points) Find a basis for the subspace from part (a).

8. The matrix  $A = \begin{bmatrix} 5 & 4 & 7 & -2 & 8 \\ 1 & 4 & k & -2 & 5 \\ 0 & 2 & -4 & -1 & 2 \\ 7 & 0 & 21 & 0 & 7 \end{bmatrix}$  reduces to  $R = \begin{bmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & -2 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ .

Let  $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4, \vec{a}_5$  be the columns of  $A$ .

- (a) (1 point) Choose a basis for  $\text{Col}(A)$  from the columns of  $A$ .  
 (b) (1 point) Given  $\text{Col}(A)$  is a subspace of  $\mathbb{R}^n$ , what is the value of  $n$ ?  
 (c) (1 point) What is  $\text{Nul}(A)$ ? *Circle one from the choices below:*

A point      A line      A plane       $\mathbb{R}^3$        $\mathbb{R}^4$        $\mathbb{R}^5$

- (d) (2 points) Solve for the unknown  $k$ .  
 (e) (1 point) Choose another basis for  $\text{Col}(A)$  from the columns of  $A$ .

9. Let  $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 1 & 1 \end{bmatrix}$  and  $\vec{b} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ .

(a) (1 point) Show that the vector  $\vec{u} = \begin{bmatrix} 3 \\ -1 \\ 0 \\ 2 \end{bmatrix}$  satisfies  $A\vec{u} = \vec{b}$ .

- (b) (3 points) Choose a basis for  $\text{Nul}(A)$ .  
 (c) (2 points) **Use results from part (a) and (b)** to find the general solution of the nonhomogeneous system  $A\vec{x} = \vec{b}$ .

10. (1 point) Suppose  $A^T$  is a  $7 \times 4$  matrix and that the nullity of  $A$  is 4. What is the dimension of  $\text{Col}(A)$ ?

11. (5 points) An economy consists of two industries: plastic and oil. The production of \$1 worth of plastic requires 30¢ of plastic and 60¢ of oil. The production \$1 of oil requires 20¢ of plastic and 80¢ of oil.

- (a) Which, if any, of the industries are profitable? Justify.  
 (b) Is the economy productive? Justify.  
 (c) If there is an external demand for \$2 million of plastic and \$5 million of oil, how much of each industry should be produced to meet it?

12. (5 points) Lisa eats breakfast every day, and has either oatmeal or a smoothie. She also for some reason always keeps track of what she eats every day, and notices a pattern for her breakfast. If she has oatmeal for breakfast on a given day, there's a  $\frac{1}{3}$  chance that she'll have oatmeal again the next morning. If she has a smoothie however, then there's a  $\frac{2}{5}$  chance that she'll have another smoothie the next morning.

- (a) Write down the transition matrix  $P$  associated with this Markov chain.  
 (b) Find the steady state vector  $\vec{q}$ . Give your answers as fractions. Interpret.

**13.** (5 points) A company manufactures 3 types of tents: small tents, medium-sized tents and large tents. Each small tent requires 2 hours of cutting and 4 hours of sewing. Each medium-sized tent requires 3 hours of cutting and 5 hours of sewing. Each large tent requires 4 hours of cutting and 7 hours of sewing. The company has 333 hours of cutting and 586 hours of sewing available each month. Market research also shows that the company can sell a maximum of 120 tents in a month. The company's profit margin is \$50 for small tents, \$60 for medium-sized tents and \$40 for large tents.

- (a) Define your variables and set up the linear program that would be used to determine the number of tents of each type the company should make to maximize their monthly profit. When defining  $x_1$  etc., please respect the order in which things were named in the question. Also, please maintain the order of the given constraints. **Do not solve.**

Below is the final simplex table corresponding to this linear program.

Use it to answer the questions in the parts that follow.

$$\left[ \begin{array}{c|cccc|ccc|c} 1 & 0 & 0 & 30 & 10 & 0 & 30 & 6930 \\ \hline 0 & 0 & 1 & 2 & 1 & 0 & -2 & 93 \\ 0 & 0 & 0 & 1 & -1 & 1 & -2 & 13 \\ 0 & 1 & 0 & -1 & -1 & 0 & 3 & 27 \end{array} \right]$$

- (b) How many tents of each type should the company make to maximize their profit?
- (c) What is the maximum profit?
- (d) When profit is maximized, how many cutting hours and sewing hours are unused?

## ANSWERS

1.  $(2, 0, 12), (0, 1, 15)$

2.  $\begin{bmatrix} -4 & -4 \\ 7 & 10 \end{bmatrix}$

3. (a) Perpendicular, since  $\vec{u} \cdot \vec{v} = 0$

(b)  $\vec{x} = (3, 0, 1) + t(-4, 5, 1)$

(c)  $k \neq 3$

4.  $\det(U) = 0$ , so L.D.

5. (a) F (The nullity of  $A^T$  is 0, since the rank of  $A^T = 4$ ).

(b) T ( $\det(2A^{-1}) = 2^4 \frac{1}{\det(A)} = 2^4(\frac{1}{2}) = 2^3 = 8$ ).

(c) F ( $\det(A) = \det(A^T) \neq 0$  so  $A$  IS invertible.)

6. 2

7. (a)  $x - 2y + z = 0$

(b)  $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$  answers may vary.

8. (a)  $\{\vec{a}_1, \vec{a}_2, \vec{a}_5\}$

(b) 4

(c) A plane

(d)  $k = -5$

(e)  $\{\vec{a}_1, \vec{a}_4, \vec{a}_5\}$  is one possibility.

9. (a)  $A$  times  $\vec{u}$  equals  $\vec{b}$ .

(b)  $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$

(c)  $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \\ 2 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -3 \\ 1 \end{bmatrix}$

10. 3

11. (a) Plastic only

(b) Yes because  $(I - C)^{-1}$  is non-negative.

(c) \$70 million of plastic and \$235 million of oil

12. (a)  $P = \begin{bmatrix} 1 & 3 \\ \frac{3}{3} & \frac{3}{5} \end{bmatrix}$

(b)  $(\frac{9}{19}, \frac{10}{19})$

In the long run, Lisa will eat oatmeal 9 out of every 19 mornings, whereas she will eat a smoothie 10 out of every 19 mornings.

13. (a)  $x_1$  = the number of small tents,  $x_2$  = the number of medium-sized tents,  $x_3$  = the number of large tents

$$\text{Maximize } z = 50x_1 + 60x_2 + 40x_3, \text{ subject to } \begin{cases} 2x_1 + 3x_2 + 4x_3 \leq 333 \\ 4x_1 + 5x_2 + 7x_3 \leq 586 \\ x_1 + x_2 + x_3 \leq 120 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

- (b) 27 small tents, 93 medium-sized tents, no large tents

(c) \$6930

(d) no cutting hours, 13 sewing hours remain.