

$$1. \text{ Given } A = \begin{bmatrix} 3 & 4 & 8 & 0 & 5 & 3 \\ 3 & 2 & -2 & 0 & 2 & 13 \\ 0 & -2 & -10 & 0 & -3 & 10 \\ 6 & 4 & -4 & 0 & 9 & 16 \end{bmatrix} \sim R = \begin{bmatrix} 1 & 0 & -4 & 0 & 0 & 7 \\ 0 & 1 & 5 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

- Write the last column of  $A$  as a linear combination of the first five columns of  $A$ .
- Find a basis for  $\text{Col}(A)$ .
- Find a basis for  $\text{Row}(A)$ .
- Find a basis for  $\text{Nul}(A)$ .
- What is the dimension of  $\text{Nul}(A^T)$ ?

2. Let  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the linear transformation that first performs the horizontal shear that sends the vector  $\mathbf{e}_2$  to  $-\mathbf{e}_1 + \mathbf{e}_2$  (leaving  $\mathbf{e}_1$  unchanged), then reflects points through the  $y$ -axis, and then rotates points about the origin by  $\frac{\pi}{3}$  radians in the *clockwise* direction. Find the standard matrix for  $T$ .

$$3. \text{ Given } A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \text{ and } \det(A) = -3. \text{ Evaluate the following.}$$

$$(a) \det \left( \begin{bmatrix} a/5 & b/5 & c/5 \\ g & h & i \\ a + 4d & b + 4e & c + 4f \end{bmatrix} \right)$$

$$(b) \det(2A^2A^T)$$

$$(c) \det(A^{-1} + \text{adj}(A))$$

$$4. \text{ Let } A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 6 \\ 4 & 6 & 1 \end{bmatrix}.$$

- Perform exactly one elementary row operation on  $A$  to get a *symmetric* matrix. (Call it  $B$ )  
(Reminder: Matrix  $B$  is symmetric if  $B = B^T$ )
- Write  $B$  as the product,  $EA$ , where  $E$  is an elementary matrix.

5. Given  $X, B$ , and  $C$  are  $n \times n$  matrices, solve the following equation for  $X$ . Assume any necessary matrices to be invertible.

$$(3X^{-1}B)^{-1} = C(X + B)$$

$$6. \text{ Let } A = \begin{bmatrix} 1 & -2 & -4 \\ -2 & 6 & 10 \\ -1 & 2 & 3 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}.$$

- Determine  $A^{-1}$ .
- Use your answer in part (a) to solve the matrix equation  $A\mathbf{x} = \mathbf{b}$ .

7. Determine a specific  $3 \times 4$  matrix  $A$  that meets the following two conditions. Show the conditions are satisfied.

- The dimension of  $\text{Nul}(A)$  is equal to the rank of  $A$ .

- $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \in \text{Nul}(A)$

8. Given  $\mathcal{S} = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a^2 = d^2 \right\}$ .

- Does  $\mathcal{S}$  contain the  $2 \times 2$  zero matrix?
- Is  $\mathcal{S}$  closed under vector addition? Justify.
- Is  $\mathcal{S}$  closed under scalar multiplication? Justify.

9. Given  $T : \mathbb{R}^3 \rightarrow \mathbb{R}$  defined by  $T(\mathbf{u}) = \mathbf{u} \cdot \mathbf{u}$ . Use specific vectors to show  $T$  is not a linear transformation.

10. Find the point on the line  $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$  closest to the point  $(1, -3, -10)$ .

11. Find a non-zero vector which is orthogonal to the plane that contains the points  $P(5, 2, 3)$ ,  $Q(6, 0, 3)$ , and  $R(7, 5, 1)$ .

12. Let  $\mathcal{P}_1$  be the plane with equation  $2x - y + z = 3$  and let  $\mathcal{P}_2$  be the plane with equation  $x + y + 2z = 1$ .

- Find a parametric vector equation for the line of intersection of planes  $\mathcal{P}_1$  and  $\mathcal{P}_2$ .
- For what values  $a$  and  $b$  does the point  $(11, a, b)$  lie on this line of intersection?
- Find an equation for the plane  $\mathcal{P}_3$  that is parallel to  $\mathcal{P}_2$  and passes through  $(1, 2, 3)$ .
- Find the distance between  $\mathcal{P}_2$  and  $\mathcal{P}_3$ .

13. Let  $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ ,  $\mathbf{v} = \begin{bmatrix} 2 \\ -4 \\ 8 \end{bmatrix}$ , and  $\mathbf{w} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$ .

- Find the volume of the parallelepiped formed by vectors  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$ .
- True or false:  $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$  forms a basis for  $\mathbb{R}^3$ .

(c) Use your answer in part (a) to find the determinant of the matrix  $\begin{bmatrix} 0 & 0 & 0 & 3 \\ 2 & 2 & 0 & 0 \\ -4 & 1 & 3 & 0 \\ 8 & 3 & 1 & 0 \end{bmatrix}$ .

14. Fill in the blank with the word *must*, *might*, or *cannot*, as appropriate.

- A system of linear equations with more equations than variables \_\_\_\_\_ have a unique solution.
- If  $\{\mathbf{u}, \mathbf{v}\}$  and  $\{\mathbf{u}, \mathbf{w}\}$  are both linearly independent sets, then  $\{\mathbf{v}, \mathbf{w}\}$  \_\_\_\_\_ be linearly independent.

(c) If  $A$  is a  $3 \times 3$  matrix such that  $A^3 = I$ , then  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  \_\_\_\_\_ form a basis for  $\text{Col}(A)$ .

(d) If  $\mathbf{u} \cdot \mathbf{v} \neq 0$ , then vectors  $\mathbf{u}$  and  $\mathbf{v}$  \_\_\_\_\_ be parallel.

### Answers

$$1. \text{ (a) } \begin{bmatrix} 3 \\ 13 \\ 10 \\ 16 \end{bmatrix} = 7 \begin{bmatrix} 3 \\ 3 \\ 0 \\ 6 \end{bmatrix} - 2 \begin{bmatrix} 4 \\ 2 \\ -2 \\ 4 \end{bmatrix} + 0 \begin{bmatrix} 8 \\ -2 \\ -10 \\ -4 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} 5 \\ 2 \\ -3 \\ 9 \end{bmatrix} \quad \text{(b) } \left\{ \begin{bmatrix} 3 \\ 3 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ -3 \\ 9 \end{bmatrix} \right\}$$

(c)  $\{[1 \ 0 \ -4 \ 0 \ 0 \ 7], [0 \ 1 \ 5 \ 0 \ 0 \ -2], [0 \ 0 \ 0 \ 0 \ 1 \ -2]\}$  (other answers exist)

$$\text{(d) } \left\{ \begin{bmatrix} 4 \\ -5 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 2 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\} \quad \text{(e) } 1$$

$$2. \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 & 1 + \sqrt{3} \\ \sqrt{3} & 1 - \sqrt{3} \end{bmatrix}$$

$$3. \text{ (a) } \frac{12}{5} \quad \text{(b) } -216 \quad \text{(c) } \frac{8}{3}$$

$$4. \text{ (a) } \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 6 \\ 4 & 6 & 1 \end{bmatrix} \xrightarrow{-2R_1 + R_2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 6 \\ 0 & 6 & 1 \end{bmatrix} = B \quad \text{(b) } B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 6 \\ 4 & 6 & 1 \end{bmatrix}$$

$$5. X = 3(B^{-1} - 3C)^{-1}CB$$

$$6. \text{ (a) } A^{-1} = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1/2 & 1 \\ -1 & 0 & -1 \end{bmatrix} \quad \text{(b) } \mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1/2 & 1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \\ -5 \end{bmatrix}$$

$$7. A = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(other answers exist)}$$

8. (a) Yes (b)  $\mathcal{S}$  is not closed under addition (c)  $\mathcal{S}$  is closed under scalar multiplication

$$9. \text{ Let } \mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u} + \mathbf{v} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}; T(\mathbf{u}) = 1, T(\mathbf{v}) = 4 \rightarrow T(\mathbf{u}) + T(\mathbf{v}) = 5 \text{ however } T(\mathbf{u} + \mathbf{v}) = 9$$

10.  $(3, -6, -6)$

11.  $(4, 2, 7)$

$$12. \text{ (a) } \mathbf{x} = \begin{bmatrix} 4/3 \\ -1/3 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \quad \text{(b) } \begin{matrix} a = 28/3 \\ b = -29/3 \end{matrix} \quad \text{(c) } x + y + 2z = 9 \quad \text{(d) } \frac{4}{3}\sqrt{6}$$

13. (a)  $40u^3$  (b) True (c)  $-120$

14. (a) might (b) might (c) must (d) might