Please Note: This was a 2 hour exam, which did not test course content from Chapters 10 and 11 of the textbook (Stewart Calculus Early Transcendentals 8th edition).

1. Identify and sketch each of the following:

(a) $x^2 + y^2 + z^2 = 2y$

(b) The level surface of $w = f(x, y, z) = \frac{x^2}{9} - y^2 - z^2$ for $w = 1$.

(c) $\rho = 4\csc(\phi)\cot(\phi)$

2. A curve is defined by $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), t\rangle$.

(a) Find the velocity and acceleration vectors: $\mathbf{v}(t)$ and $\mathbf{a}(t)$.

(b) Find the unit tangent and unit normal vectors at $t = 0$.

(c) Find the curvature κ at $t = 0$.

3. Is the following function continuous at the origin? Be sure to properly justify your answer.

$$f(x, y) = \begin{cases} \dfrac{5x^2y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

4. Given the implicit relation $x^2yz = \dfrac{x\sin(z)}{y^2} + 1$, find $\dfrac{\partial z}{\partial x}$.

5. Given $f(x, y) = x^2 \ln\left(\dfrac{y}{x^2}\right)$, find:

(a) $\dfrac{\partial^2 f}{\partial x \partial y}$,

(b) $\dfrac{\partial z}{\partial r}$, where $z = f(x, y)$, $x = r^2 + s^2$, and $y = 2rs$.

6. If $f(u, v, w)$ is a differentiable function and $F = f(x - y, y - z, z - x)$, show that $\dfrac{\partial F}{\partial x} + \dfrac{\partial F}{\partial y} + \dfrac{\partial F}{\partial z} = 0$.

7. Given the level surface $\mathcal{S}$: $f(x, y, z) = x - y^3 - z^2 = 3$ and the point $P(-4, -2, 1)$,

(a) find the equation of the tangent plane to $\mathcal{S}$ at the point P,

(b) find the directional derivative of f at P in the direction of $v = \langle 3, 6, -2\rangle$,

(c) find the maximum rate of change in f at P,

(d) show that $\mathbf{r}(t) = \langle 2t^5 + t^4 - 7, t^2 - t - 2, t\rangle$ is tangent to the surface $\mathcal{S}$ at P.

8. Find and classify the critical points of $f(x, y) = x^3 + 3xy^2 + 3y^2 - 15x + 2$.

9. Evaluate $\displaystyle\iint_D x \, \mathrm{d}A$, where D is the region bounded by the line $y = x + 1$ and the parabola $y = \frac{1}{2}\left(x^2 - 6\right)$.

10. Evaluate the following integrals, changing the order of integration or coordinate system as needed.

(a) $\displaystyle\int_0^9 \int_{\sqrt{x}}^3 xy \sin\left(y^6\right) \, \mathrm{d}y \, \mathrm{d}x$

(b) $\displaystyle\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} e^{x^2+y^2} \, \mathrm{d}x \, \mathrm{d}y$

11. Rewrite the integral $\displaystyle\int_{-1}^1 \int_{x^2}^1 \int_0^{1-y} x \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x$ in the order $\mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$.

12. Sketch the solid region $\mathcal{S}$ between the cone $z = \sqrt{x^2+y^2}$ and the xy plane, inside the cylinder $x^2+y^2=1$.

(a) Set up the triple integral necessary to find the volume of $\mathcal{S}$:

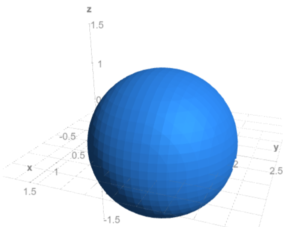
(i) using cylindrical coordinates

(ii) using spherical coordinates

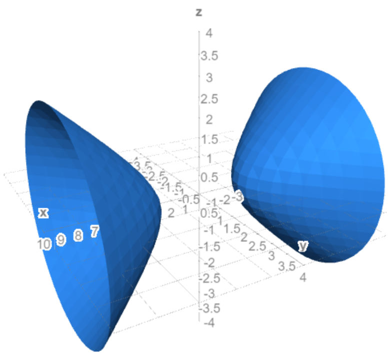
(b) Evaluate *one* of these integrals to determine the volume of $\mathcal{S}$.

Answers

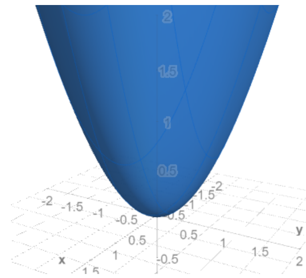
1. (a) Sphere of radius 1, centered at $(0, 1, 0)$



(b) Hyperboloid in two sheets, intersecting the x-axis at $(\pm 3, 0, 0)$



(c) Elliptical (circular) paraboloid



(a) $\mathbf{v}(t) = \left\langle -2\sin(t), 2\cos(t), t \right\rangle, \quad \mathbf{a}(t) = \left\langle -2\cos(t), -2\sin(t), 0 \right\rangle$

(b) $\mathbf{T}(t) = \dfrac{1}{\sqrt{5}} \langle 0, 2, 1 \rangle, \quad \mathbf{N}(t) = \langle -1, 0, 0 \rangle$

(c) $\dfrac{2}{5}$

2.

Notice that:

$$0 \leq x^2 \leq x^2 + y^2$$

$$0 \leq \frac{x^2}{x^2+y^2} \leq 1$$

$$0 \leq \frac{5x^2|y|}{x^2+y^2} \leq 5|y|$$

take the limit in question:

$$\lim_{(x,y)\to(0,0)} 0 \leq \lim_{(x,y)\to(0,0)} \frac{5x^2|y|}{x^2+y^2} \leq \lim_{(x,y)\to(0,0)} 5|y|$$

$$0 \leq \lim_{(x,y)\to(0,0)} \frac{5x^2|y|}{x^2+y^2} \leq 0$$

$$0 \leq \lim_{(x,y)\to(0,0)} \left| \frac{5x^2 y}{x^2+y^2} \right| \leq 0$$

Therefore $\displaystyle \lim_{(x,y)\to(0,0)} \left|\frac{5x^2y}{x^2+y^2}\right| = \left|\lim_{(x,y)\to(0,0)} \frac{5x^2y}{x^2+y^2}\right| = 0$, so $\displaystyle \lim_{(x,y)\to(0,0)} \frac{5x^2y}{x^2+y^2} = 0$.

3. $-\dfrac{F_x}{F_z} = \dfrac{\sin(z) - 2xy^3z}{x^2y^3 - x\cos(z)}$

4. (a) $\dfrac{2x}{y}$

   (b) $4xr\left(\ln\left(\dfrac{y}{x^2}\right) - 1\right) + \dfrac{2x^2s}{y}$

5. Using the chain rule,

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} = \left(\frac{\partial F}{\partial u} - \frac{\partial F}{\partial w}\right) + \left(-\frac{\partial F}{\partial u} + \frac{\partial F}{\partial v}\right) + \left(-\frac{\partial F}{\partial v} + \frac{\partial F}{\partial w}\right) = 0$$

6. (a) $x - 12y - 2z = 18$

   (b) $\dfrac{-65}{7}$

   (c) $\sqrt{149}$

   (d) Note that $(-4,-2,1) = \mathbf{r}(1)$, and show that $r'(1) \circ \nabla f(-4,-2,1) = 0$.

7. Local minimum at $(\sqrt{5}, 0)$, local maximum at $(-\sqrt{5}, 0)$, saddle points at $(1, \pm 2)$.

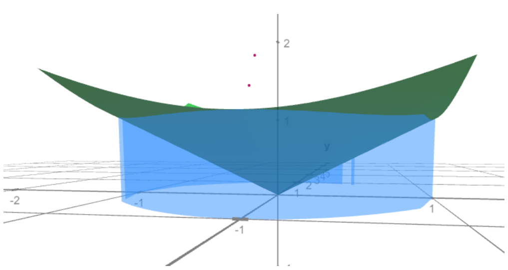
8. $\dfrac{176}{3}$

9. $\dfrac{1-\cos(729)}{12}$

10. $\dfrac{\pi}{8}\left(e^4 - 1\right)$

11. $\displaystyle \int_0^1 \int_0^{1-z} \int_{-\sqrt{y}}^{\sqrt{y}} x \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$

12. The region to be sketched is shown in blue:



   (a) (i) $\displaystyle \int_0^{2\pi} \int_0^1 \int_0^r r \, \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}\theta$ (ii) $\displaystyle \int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\csc(\phi)} \rho^2 \sin(\phi) \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\theta$

   (b) $\dfrac{2\pi}{3}$ units^3