1. Solve each of the following systems or show that it is inconsistent.

(a) $\begin{cases} 2x + y - 3z = 4 \\ 2x + 8z = 16 \\ 5x + y + 9z = 21 \end{cases}$

(b) $\begin{cases} 4x_1 + 5x_2 + 2x_3 = 18 \\ 3x_1 + 4x_2 + x_3 = 11 \\ 2x_1 + 5x_2 - 3x_3 = -12 \\ x_2 - 2x_3 = -10 \end{cases}$

2. Given the matrix representation of a system as $\left[\begin{array}{ccc|c} -1 & 2 & 1 & 2 \\ 3 & -5 & -1 & -2 \\ 0 & 2 & 2h+5 & k+4 \end{array}\right]$, find the value(s) of h and k, if any, for which the system has

(a) No solutions.

(b) Infinitely many solutions.

(c) A unique solution.

3. Given that $A = \begin{bmatrix} 1 & 0 \\ -2 & 3 \\ -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 5 & -1 & -2 \\ 1 & 2 & 4 \\ -2 & 0 & 3 \end{bmatrix}$, and $C = \begin{bmatrix} -2 & 6 & -3 \\ 5 & 4 & -1 \end{bmatrix}$, find the following, if possible.

(a) A^2

(b) BA

(c) $2A + C^T$

4. Given that $D = \begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix}$, $E = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$, find a matrix X such that:

$$(XD)^T = E$$

5. Given that $\begin{vmatrix} 4 & 7 & -2 \\ 2 & -5 & -2 \\ -3 & 8 & 6 \end{vmatrix} = -100$,

(a) evaluate $\begin{vmatrix} 0 & 0 & 0 & 2 \\ 4 & 7 & -2 & 0 \\ 2 & -5 & -2 & 0 \\ -3 & 8 & 6 & 0 \end{vmatrix}$

(b) use Cramer's Rule to solve for y **only** given the following system of equations: $\begin{cases} 4x + 7y - 2z = 3 \\ 2x - 5y - 2z = -1 \\ -3x + 8y + 6z = 7 \end{cases}$

6. Let A, B and C be 5×5 matrices such that $\det(A) = -4$, $\det(B) = \frac{1}{5}$ and $\det(C) = 3$. Find, if possible:

(a) $\det(A^T \cdot B^{-1} \cdot C)$

(b) $\det(C - 3I)$

(c) $\det\left((2B)^{-1}\right)$

7. Consider the matrix $\begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & -2 \\ 3 & 1 & -3 \end{bmatrix}$.

(a) Find adj(A).

(b) Calculate $A \cdot \text{adj}(A)$.

(c) Use your previous work to find A^{-1}.

8. Given the points $A = (1, 2, 1)$, $B = (-1, 5, 7)$ and $C = (4, 2, 0)$,

(a) find the magnitude (length) of the vector $\overrightarrow{AB}$

(b) find **both** unit vectors parallel to $\overrightarrow{AB}$.

(c) find a vector equation for the plane through the points A, B and C.

(d) find an equation in general form $(ax + by + cz = d)$ for the plane through the points A, B and C.

(e) find an equation of the line passing through the point A and perpendicular to the plane $4x - y + 2z = 7$.

9. Is the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix} t + \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$ parallel, perpendicular or neither to the plane $-x + 3y + z = 12$?

10. Given that $z = -2x + y$ is a subspace of $\mathbb{R}^3$,

(a) find a basis for the above subspace.

(b) what is its dimension?

11. Consider the set of vectors $\overrightarrow{u} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$, $\overrightarrow{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, and $\overrightarrow{w} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}$.

(a) Write the vector $\overrightarrow{b} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$ as a linear combination of the the vectors $\overrightarrow{u}, \overrightarrow{v}$ and $\overrightarrow{w}$, if possible.

(b) Is the set $\{\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}\}$ linearly independent or linearly dependent?

(c) Describe span$\{\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}\}$: Is it a point in $\mathbb{R}^3$, a line in $\mathbb{R}^3$, a plane in $\mathbb{R}^3$, or all of $\mathbb{R}^3$? **Justify**.

(d) Provide a basis for span$\{\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}\}$.

(e) What is the dimension of span$\{\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}\}$?

12. Suppose $A = \begin{bmatrix} 3 & -4 & 10 & -1 & -2 \\ 6 & -7 & 19 & -2 & -4 \\ -2 & 4 & -8 & 1 & 3 \\ 3 & 2 & 4 & 0 & 3 \end{bmatrix}$ which reduces to $R = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$,

Let $\overrightarrow{a_1}$ be the first column of A, $\overrightarrow{a_2}$ be the second column of A ...

(a) Find a basis for Col(A).

(b) Write $\overrightarrow{a_3}$ as a linear combination of $\overrightarrow{a_1}, \overrightarrow{a_2}$ and $\overrightarrow{a_4}$ if possible.

(c) Write $\overrightarrow{a_5}$ as a linear combination of $\overrightarrow{a_1}, \overrightarrow{a_2}$ and $\overrightarrow{a_3}$ if possible.

(d) Find a basis for Null(A).

(e) State the following sets as linearly independent or dependent.

i. $\{\overrightarrow{a_1}, \overrightarrow{a_2}, \overrightarrow{a_3}\}$

ii. $\{\overrightarrow{a_2}, \overrightarrow{a_4}, \overrightarrow{a_5}\}$

iii. $\{\overrightarrow{a_1}, \overrightarrow{0}\}$

13. If B is a 4×6 matrix such that when reduced would have 3 pivots, find

(a) the Rank of B.

(b) the Nullity of B^T.

(c) the number of solutions to $B\overrightarrow{x} = \overrightarrow{0}$.

14. Complete the following sentences with the word MUST, MIGHT, or CANNOT, as appropriate:

(a) Two planes with parallel normal vectors ____________ intersect.

(b) If the set of vectors $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}\}$ is linearly dependent then there ____________ be a linear combination of $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ to make $\overrightarrow{v_3}$.

(c) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 4 & 6 \end{bmatrix}$ then $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ ____________ be in Null(A).

(d) If B is a 3×3 matrix such that Null(B) is a line then Col(B) ____________ be a plane.

15. An economy has two industries: Math and Happiness. To produce \$1 of Math requires \$0.30 of Math and \$0.50 of Happiness. To produce \$1 of Happiness \$0.70 of Math and \$0.40 of Happiness.

(a) Find a consumption matrix C associated with this economy.

(b) Which industries, if any, are profitable? Justify your answer.

(c) Given an external demand for \$1890 of Math and \$945 of Happiness, how much of each industry should be produced to meet it?

(d) Find the internal consumption when the demand is met.

16. On a dystopic tropical island, a group of hostages are forced to compete each day in The Octopus Games. It has been established over time that if a player wins in a game one day, there is a 60 % chance that they will again win the game played the next day. However, if the player loses in a game one day, there is a 80 % chance that they will lose again at the next day's game.
(Note that unlike in the Squid Games, players are not eliminated when they lose. *Whew!*)

(a) What is a transition matrix P that describes this Markov chain?

(b) On one Tuesday, 25 % of the players won while 75 % lost. What proportion will win at Thursday's game?

(c) Find a steady-state vector $\overrightarrow{q}$ for this Markov chain. Use fractions in your answer.

(d) In the long-run, what proportion of players will lose?

These tables may come in handy:

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26

a	1	3	5	7	9	11	15	17	19	21	23	25
a^{-1}	1	9	21	15	3	19	7	23	11	5	17	25

17. The answer to the riddle was encrypted using the matrix: $B = \begin{bmatrix} 7 & 2 \\ 3 & 3 \end{bmatrix}$

(a) Find the decryption matrix B^{-1} and verify your answer using matrix multiplication.

(b) Decode the ciphertext below (in its entirety) to reveal the answer to:
The most watched Netflix show by aquatic animals is: _______ Game

TIOPVF

Solutions:

1. (a) No solution

(b) (5,-2,4)

2. (a) $h = -1/2, k \neq 4$

(b) $h = -1/2, k = 4$

(c) $h \neq -1/2$

3. (a) does not exist

(b) $\begin{bmatrix} 9 & -7 \\ -7 & 14 \\ -5 & 6 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 5 \\ 2 & 10 \\ -5 & 3 \end{bmatrix}$

4. $X = \dfrac{1}{11}\begin{bmatrix} 4 & 5 \\ 5 & 9 \end{bmatrix}$

5. (a) 200

(b) $y = \dfrac{-8}{-100} = \dfrac{2}{25}$

6. (a) -60

(b) Can't tell

(c) $\dfrac{5}{32}$

7. (a) adj(A)=$\begin{bmatrix} 2 & -3 & 2 \\ 0 & -3 & 2 \\ 2 & -4 & 2 \end{bmatrix}$.

(b) A adj(A)=$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(c) $A^{-1} = \dfrac{1}{2}\begin{bmatrix} 2 & -3 & 2 \\ 0 & -3 & 2 \\ 2 & -4 & 2 \end{bmatrix}$

8. (a) 7

(b) $\dfrac{1}{7}\langle -2, 3, 6\rangle$ and $\dfrac{-1}{7}\langle -2, 3, 6\rangle$

(c) $(x, y, z) = (-2, 3, 6)t + (5, -3, -7)s + (1, 2, 1)$ or equivalent

(d) $-3x + 16y - 9z = 20$

(e) $(x, y, z) = (4, -1, 2)t + (1, 2, 1)$ or equivalent

9. Parallel

10. (a) $\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

(b) 2

11. (a) Not possible

(b) Linearly dependant

(c) A Plane

(d) $\{\overrightarrow{u}, \overrightarrow{v}\}$.

(e) 2

12. (a) $\left\{ \begin{bmatrix} 3 \\ 6 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ -7 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}$ or equivalent

(b) $\overrightarrow{a_3} = 2\overrightarrow{a_1} - \overrightarrow{a_2}$

(c) Not possible.

(d) $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ -5 \\ 1 \end{bmatrix} \right\}$

(e) i. Linearly Dependant

ii. Linearly Independant

iii. Linearly Dependant

13. (a) 3

(b) 1

(c) infinite

14. (a) MIGHT

(b) MIGHT

(c) MUST

(d) MUST

15. (a) $C = \begin{bmatrix} 0.3 & 0.7 \\ 0.5 & 0.4 \end{bmatrix}$

(b) Math

(c) \$ 25650 of Math, \$ 22950 of Happiness

(d) \$ 23760 of Math, \$ 22005 of Happiness

16. (a) $\begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}$

(b) 32 %

(c) $\mathbf{q} = (\frac{1}{3}, \frac{2}{3})$

(d) $\frac{2}{3}$

17. (a) $B^{-1} = \begin{bmatrix} 21 & 12 \\ 5 & 23 \end{bmatrix}$

(b) HUMAN