1. Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 4 & -1 \\ 2 & 2 & 2 & 2 \\ 3 & 2 & -1 & 5 \end{bmatrix}$.

(a) (3 points) Write the solution to $A\mathbf{x} = \mathbf{0}$ in parametric vector form.

(b) (2 points) Find a vector in $\text{Nul}(A)$ that has a 4 in its first entry.

(c) (1 point) Which column of A cannot be written as a linear combination of the other columns of A?

(d) (1 point) Give a basis for $\text{Col}(A)$.

2. (3 points) Solve for a,b and c in the matrix multiplication below.

$\begin{bmatrix} a & -2 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} b & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ c & -4 \end{bmatrix}$.

3. (3 points) Given that A and B are invertible with B also being symmetric, solve for matrix X. Your answer should be expressed as a single term.

$$B^T AX - A = (B - I)(B + I)A$$

4. Given that A, B and C are 2×2 matrices such that $\det(A) = -2$, $\det(B) = -5$, and $\text{rank}(C) = 1$, evaluate the following determinants (or explain why there is not enough information.)

(a) (2 points) $\det(3AB^{-1})$

(b) (2 points) $\det(\text{adj}(A^T))$

(c) (2 points) $\det(AC + BC)$

5. (3 points) You are given the following transformations.

Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the shear transformation such that $S\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$.

Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation through $\frac{\pi}{2}$ radians. (*Note: This is counterclockwise.*)

Find the standard matrix of the composite transformation $R \circ S$.

6. Let $A = \begin{bmatrix} 2 & 8 & 8 & 8 \\ 2 & 4 & 1 & 1 \\ 0 & 4 & 1 & 1 \\ 0 & 4 & 0 & 1 \end{bmatrix}$.

(a) (3 points) Find $\det(A)$.

(b) (2 points) Solve the system $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ for x_2 **only**.

7. Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 6 \\ 2 & 4 & 6 \end{bmatrix}$

(a) (3 points) Find A^{-1}.

(b) (2 points) Use your answer in part (a) to solve for the 1×3 matrix X in the following equation: $XA = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}$

(c) (2 points) Find the elementary matrix E such that $EA = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix}$.

8. Let $H = \{A \in \mathbb{M}_{2\times 2} : A \text{ is non-invertible}\}$.

(a) (2 points) List two non-zero vectors in H.

(b) (2 points) Is H closed under scalar multiplication? Justify your answer.

(c) (2 points) Is H closed under addition? Justify your answer.

(d) (1 point) Is H a subspace of $\mathbb{M}_{2\times 2}$? Justify your answer.

9. You are given the following vectors in $\mathbb{P}_3$.

$p(x) = x^3 + 2x^2 - 1$

$q(x) = 2x^3 + 3x^2 - x + 2$

$r(x) = -2x^3 - x^2 + 3x - 10$

(a) (3 points) Is the set $\{p(x), q(x), r(x)\}$ linearly independent or linearly dependent?

(b) (1 point) What is the dimension of the span of $\{p(x), q(x), r(x)\}$?

(c) (1 point) What is the dimension of $\mathbb{P}_3$?

10. (4 points) Let $T : \mathbb{R}^3 \to \mathbb{R}$ be defined by $T(\mathbf{x}) = \mathbf{x} \cdot \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}$. Find a basis for the kernel of T.

11. Complete each of the following sentences with the word "must", "might" or "cannot".

(a) (1 point) If $T : \mathbb{R}^3 \to \mathbb{R}^4$ is a linear transformation, then T ____________ be one-to-one.

(b) (1 point) A^TA ____________ be a symmetric matrix.

(c) (1 point) If $B^2 = B$, then B ____________ be invertible.

(d) (1 point) For a square matrix C, the dimension of Nul(C) ____________ equal the determinant of C.

12. Let $\mathbf{u} = \begin{bmatrix} h \\ -1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}$

(a) (2 points) Find all values of h so that $\mathbf{u}$ is orthogonal to $\mathbf{v}$.

(b) (2 points) Find all values of h so that $\mathbf{u}$ is parallel to $\mathbf{v}$.

(c) (2 points) Find all values of h so that $||\mathbf{u}|| = 3$.

(d) (2 points) Let $h = 4$ and find the *cosine* of the angle between $\mathbf{u}$ and $\mathbf{v}$.

13. You are given the points $A(1, 1, 1)$, $B(2, 4, 3)$ and $C(3, 2, 2)$.

(a) (2 points) Write an equation of the line that contains C and is parallel to the line that goes through A and B.

(b) (3 points) Write a normal equation of the plane that contains A, B and C. (Recall that the normal form is $ax + by + cz = d$.)

(c) (2 points) Calculate the area of the triangle ABC.

14. (4 points) You are given the point $A(-2, 3, -8)$ and the line $\mathcal{L}$ given by $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$. Find the point on $\mathcal{L}$ that is closest to A.

Answers

1. (a) $\mathbf{x} = t \begin{bmatrix} 3 \\ -4 \\ 1 \\ 0 \end{bmatrix}, t \in \mathbb{R}$; (b) $\begin{bmatrix} 4 \\ -\frac{16}{3} \\ \frac{4}{3} \\ 0 \end{bmatrix}$ (c) The fourth (d) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \\ 5 \end{bmatrix} \right\}$

2. $a = 1, b = 2, c = -6$

3. $A^{-1}BA$

4. (a) $\frac{18}{5}$, (b) -2, (c) 0

5. $A_{R \circ S} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 1 & 0 \end{bmatrix}$

6. (a) 48, (b) $-\frac{1}{24}$

7. (a) $A^{-1} = \begin{bmatrix} 3 & 2 & -3 \\ 0 & -1 & 1 \\ -1 & 0 & \frac{1}{2} \end{bmatrix}$, (b) $\begin{bmatrix} 1 & -1 & 1 \end{bmatrix}$, (c) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

8. (a) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ (Answers may vary.), (b) Yes. If $A \in H$, then $\det(A) = 0$. If $k \in \mathbb{R}$, then $\det(kA) = k^2 \det(A) = k^2 \cdot 0 = 0$. Therefore $kA \in H$. (c) No. Using the two vectors from part (a), notice that their sum is $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, which is invertible and therefore not in H. (d) No, since H doesn't satisfy closure under addition.

9. (a) Linearly dependent, since $r(x) = 4p(x) - 3q(x)$, (b) 2, (c) 4

10. $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} \right\}$

11. (a) might, (b) must, (c) might, (d) cannot

12. (a) 10, (b) $-\frac{1}{2}$, (c) $2, -2$, (d) $-\frac{2}{7}$

13. (a) $\mathbf{x} = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$, where $t \in R$ (Answers may vary.)

    (b) $x + 3x - 5z = -1$, (c) $\frac{\sqrt{35}}{2}$

14. $(-5, 0, -5)$