1. (2 points) Write $\dfrac{(1+i)^2}{3-4i}$ *in rectangular form.*

2. (4 points) Find all solutions to $z^4 = z$. *Give any complex answers in exponential form.*

3. (3 points) Let $z_1 = e^{\frac{11\pi}{12}i}$ and $z_2 = e^{\frac{2\pi}{3}i}$ Find $\left(\dfrac{z_1}{z_2}\right)^{22}$ *in rectangular form.*

4. (6 points) You are given two bases for the plane $x_1 + 2x_2 + 3x_3 = 0$:

$$\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ and } \mathcal{C} = \left\{ \begin{bmatrix} -7 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ -1 \end{bmatrix} \right\}$$

Further, let $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Find:

(a) The change of basis matrix from B to C

(b) $[\mathbf{x}]_{\mathcal{C}}$

(c) $\mathbf{x}$

5. (4 points) Left $A = \begin{bmatrix} 4 & 2 & 0 \\ 1 & 3 & 0 \\ 1 & 1 & 4 \end{bmatrix}$.

(a) Find the eigenvalues of A.

(b) Find a basis for the eigenspaces of the smallest eigenvalue.

6. (4 points) Given $A = \begin{bmatrix} 4 & 5 \\ -1 & 2 \end{bmatrix}$, find P and C such that $A = PCP^{-1}$ where C can be written as the product of a scaling matrix and a rotation matrix. (Provide only P and C... *not* P^{-1}, S or R.)

7. (3 points) **Short Answer**

Let $\mathbb{R}^3 = H_1 \oplus H_2$, where H_1 is a plane.

(a) What is the dimension of H_2? ____________

(b) What is the dimension of $H_1 \cap H_2$? ____________

(c) True or False: $H_1 \cup H_2$ is a vector space. ____________

8. (3 points) Complete each sentence with **Might, Must or Cannot.**

(a) If W is a subspace of $\mathbb{R}^n$, then $(W^\perp)^\perp$ ____________ equal W.

(b) Let A be a square matrix. The characteristic polynomial of A ____________ divide the minimum polynomial of A.

(c) Let A be a 5×5 matrix. $(\text{Col}A)^\perp$ ____________ equal Row$A$

9. (2 points) Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$.

The vector $\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ can uniquely be written $a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3$.

Use the fact that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal set to find a_3.

10. (3 points) You are given the following points. $(0,0)$, $(0,2)$, $(\ln 2, 5)$.

Find the coeficients for the equation $y = \beta_0 e^x + \beta_1 e^{-x}$ that best fits the data.

11. (4 points) Use the Gram-Schmidt process to find an **orthonormal** basis for the vector space spanned by S.

$$\mathcal{S} = \left\{ \begin{bmatrix} 0 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 16 \\ 8 \\ 2 \end{bmatrix} \right\}$$

12. (4 points) Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -2 & 11 \\ 2 & 1 & 2 \end{bmatrix}$.

(a) Find the QR-factorization of A. Note that columns 1 and 2 are orthogonal.

(b) B is a matrix such that $AB = \begin{bmatrix} 3 & 6 \\ 3 & 6 \\ 3 & 6 \end{bmatrix}$. Without finding B, what is RB equal to?

13. (8 points) Define the inner product $\langle X, Y \rangle = \mathrm{tr}(X^T Y)$ on the space $M_{2\times 2}$

Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 0 \\ d & e \end{bmatrix}$

(a) What is the length of A?

(b) Find $\langle A, B \rangle$.

(c) Find the angle between A and B.

(d) Find the values of d and e so that C is orthogonal to both A and B.

14. (6 points) Let $A = \begin{bmatrix} -5 & -3 \\ -3 & 3 \end{bmatrix}$.

(a) Write the quadratic form $Q(\mathbf{x})$ for A.

(b) Write an upper triangular matrix whose quadratic form is the same as that of A.

(c) What is the maximum value of Q given the restriction that $\mathbf{x}$ is a unit vector?

(d) Find a unit vector that achieves the maximum in part (c).

(e) A unit vector $\mathbf{u}$ is orthogonal to the vector in part (d). Find $Q(\mathbf{u})$.

15. (4 points) Let $A = \begin{bmatrix} 4 & -2 & -1 \\ 0 & 2 & -1 \\ -1 & 3 & 5 \end{bmatrix}$.

It is given that A has two eigenvalues $\lambda_1 = 3, \lambda_2 = 4$. For this problem, you are asked to *begin* finding the primary decomposition of A.

Find a suitable matrix P such that $A = PCP^{-1}$ and C is block diagonal. You do not have to provide C.

16. (2 points) Let A be a square matrix (not necessarily symmetric). Show that the quadratic form of A equals the quadratic form of A^T.

17. (2 points) Show that a square matrix with **orthonormal** columns must have a determinent equal to 1 or -1.

SOLUTIONS

1. $-\frac{8}{25}+\frac{6}{25}i$

2. $0, 1, e^{\frac{2\pi}{3}i}, e^{\frac{4\pi}{3}i}$

3. $-i$

4. a) $\begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}$; b) $\begin{bmatrix} -1 \\ -3 \end{bmatrix}$; c) $\begin{bmatrix} -8 \\ 1 \\ 2 \end{bmatrix}$

5. a) $2, 4, 5$; b) $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

6. $P = \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix}$ (Answers may vary.)

7. a) 1; b) 0; c) False

8. a) Must; b) Might; c) Cannot

9. $\frac{10}{21}$

10. $\beta_0 = 3, \beta_1 = -2$

11. $\left\{ \frac{1}{3}\begin{bmatrix} 0 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix} \right\}$

12. a) $Q = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{bmatrix}$, $R = \frac{1}{3}\begin{bmatrix} 3 & 0 & 9 \\ 0 & 3 & -6 \\ 0 & 0 & 3 \end{bmatrix}$; b) $RB = \begin{bmatrix} 5 & 10 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}$

13. a) $\sqrt{15}$
    b) 11
    c) $\cos^{-1} \frac{11}{\sqrt{13}\sqrt{15}}$
    d) $d = 5, e = -4$

14. a) $-5x_1^2 - 6x_1x_2 + 3x^2$
    b) $\begin{bmatrix} -5 & -6 \\ 0 & 3 \end{bmatrix}$
    c) 4
    d) $\frac{1}{\sqrt{10}}\begin{bmatrix} -1 \\ 3 \end{bmatrix}$
    e) -6

15. $P = \begin{bmatrix} -1 & -1 & 0 \\ -1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$ (Answers may vary.)

16. $Q_A(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$. Since this is a 1×1, we are free to transpose it.
    $= (\mathbf{x}^T A\mathbf{x})^T$
    $= (\mathbf{x})^T (A)^T (\mathbf{x}^T)^T$
    $= \mathbf{x}^T A^T \mathbf{x}$
    $= Q_{A^T}(\mathbf{x})$

17. Let U be a square matrix with orthonormal columns.
    Then U^T is the inverse of U.
    So, $U^T U = I$. We take the determinant of each side.
    $\det(U^T U) = \det(I)$
    $\det(U^T)\det(U) = 1$
    $\det(U)\det(U) = 1$
    $\det(U)^2 = 1$
    $\det(U) = \pm 1$