Exponential Functions

Properties of Exponents

Remember! Exponents come from repeated multiplication. When we write something like 2^4, this really means $\underbrace{2\cdot 2\cdot 2\cdot 2}_{\text{4 times}}$. If you keep that in mind, the properties of exponents should make sense. The number we are multiplying is called the **base** of the exponent. The number of times it's multiplied is the **exponent**. So the notation is $(base)^{exponent}$.

Quick Look at Properties of Exponents

1. $a^{x+y} = a^x \cdot a^y$ **Ex.** 2^{3+5} is the same as $2^3 \cdot 2^5$, since you could write both of them as $\underbrace{2\cdot 2\cdot \ldots \cdot 2}_{\text{8 times}}$
2. $a^{x-y} = \dfrac{a^x}{a^y}$ **Ex.** $\dfrac{2^{5x}}{2^{2x}}$ can be re-expressed as $2^{5x-2x} = 2^{3x}$
3. $(a^x)^y = a^{x\cdot y}$ **Ex.** $(4^2)^3$ **is** $\underbrace{(4^2)(4^2)(4^2)}_{\textbf{3 times}} = 4^{2\cdot 3}$, **Ex.** $x^{3/2} = x^{3\cdot\frac{1}{2}} = (x^3)^{1/2} = \sqrt{x^3}$

---

4. $(ab)^x = a^x \cdot b^x$ **Note:** Here the **exponents** match, instead of the bases. **Ex.** $2^x \cdot 5^x = (2\cdot 5)^x = 10^x$
5. $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$ **Note:** This is the same idea as 4, but with division. **Ex.** $\dfrac{x^3}{(2x+1)^3} = \left(\dfrac{x}{2x+1}\right)^3$

Special case: The most common exponential expression is e^x. e (≈ 2.718) is an important number that's important enough to get it's own notation, like π. You'll learn why e is so important in Cal I. For now you should know that it's around 2.7, that it's the most commonly used base for exponents, and that it's the base of the most common logarithm as well.

Exercise

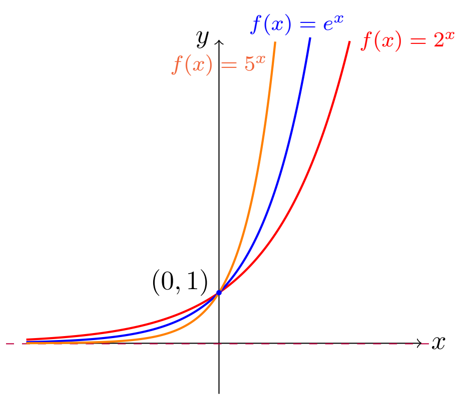
- Simplify $\dfrac{e^x 6^y}{2^y}$ as much as possible.
- Simplify $\dfrac{e^x 6^y}{6^x}$ as much as possible.
- Simplify $\dfrac{e^x e^y}{(3^x)^2}$ as much as possible.

Solution

- You can group factors with the same exponent (property 5), so we can rewrite the expression as $e^x \cdot \left(\dfrac{6}{2}\right)^y$, which then becomes $e^x \cdot 3^y$.
- Here it's the bases that batch, so using property 2 we get $e^x \cdot 6^{y-x}$.
- In the numerator we have a product of two exponents with the same base, so use property 1. In the denominator, we're taking the exponent of an exponent, so use property 3. That will give $\dfrac{e^{x+y}}{3^{2x}}$.

The Exponential Function

If I ask your calculus teacher what the **most** important thing there is for you to know about the exponential function, they will probably say that you need to know the graph! Here is the graph of the exponential function $f(x) = a^x$, for a few different choices of a. (The number chosen for a is called the **base** of the exponent.)



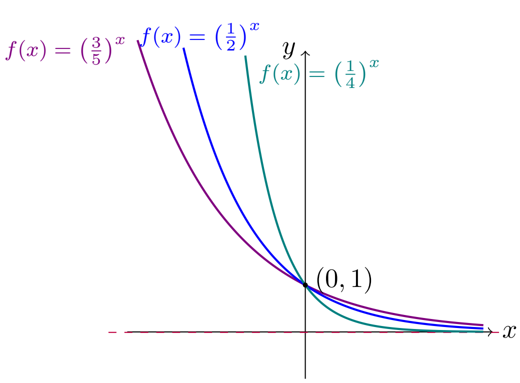
Notice that the bases chosen are all bigger than 1. When the base is greater than 1, raising it to a higher exponent gives a higher result, so the y-values grow. The bigger the base, the faster it grows.
On the other hand, raising our base to the power of a **large negative number** gives a very **small positive number**. That makes sense since

$$a^{-x} = \frac{1}{a^x}, \quad \text{so for example } 2^{-10000} = \frac{1}{2^{10000}} = \text{small positive number}$$

When a is bigger than 1:

- $a^{\text{Big positive number}} =$ Even bigger positive number
- $a^{\text{Big negative number}} =$ Small positive number
- $a^0 = 1$
- a^{anything} is always a **positive number** (never negative, and never 0)

Here's a graph of a few exponential functions with a base that is **less than** 1.



If you compare these graphs to the previous case, you'll notice that the graph is reflected arcross the $y-$axis – what used to be true for x would now be true for $-x$ instead. This makes sense since For any a we can notice that

$$a^x = a^{-1\cdot(-1x)} = \left(a^{-1}\right)^{-x} = \left(\frac{1}{a}\right)^{-x}$$

So if we take the reciprocal of any base, the behaviour at x and $-x$ are swapped.
This time, raising a to a higher exponent gives a *smaller* result, so we have a horizontal asymptote at $y = 0$ on the *right*.

For a video explanation, click here.