The Leontief Open Model

1. An economy has two industries: energy and material.
To produce \$1 of energy requires 90¢ of energy and 30¢ of material.
To produce \$1 of material requires 20¢ of energy and 30¢ of material.
   (a) Given an external demand for \$2000 of energy and \$1000 of material, how much of each industry should be produced to meet it?
   (b) Is the economy productive? Justify your answer.
   (c) Find the internal consumption when demand is met.

2. An economy has two industries: iron and steel. To produce \$1 of iron requires 20¢ of iron and 10¢ of steel. To produce \$1 of steel requires 70¢ of iron and 40¢ of steel.
   (a) Given an external demand for \$8200 of Iron and \$4100 of Steel, how much of each industry should be produced to meet it?
   (b) Which industries are profitable?

3. An economy has two industries: goods and services. To produce \$1 of goods requires 60¢ of goods and 50¢ of services. To produce \$1 of services requires 30¢ of goods and 10¢ of services.
   (a) If there is an external demand for \$6300 of goods and \$8400 of services, how much of each industry should be produced to meet it?
   (b) Is the economy productive? Justify your answer.

4. An economy has two industries: services and manufacturing. To produce \$1 of services requires 20¢ of services and 40¢ of manufacturing. To produce \$1 of manufacturing requires 30¢ of services and 10¢ of manufacturing.
   (a) If there is an external demand for \$900 of services and \$1500 of manufacturing, how much of each industry should be produced to meet it?
   (b) Find the internal consumption when demand is met.
   (c) Which industries, if any, are profitable.

5. For each of the consumption matrices below, determine which industries are profitable and whether the economy is productive.

   (a) $C = \begin{bmatrix} .8 & .3 \\ .1 & .6 \end{bmatrix}$ (b) $C = \begin{bmatrix} .8 & .1 \\ .3 & .6 \end{bmatrix}$ (c) $C = \begin{bmatrix} .8 & .1 \\ .9 & .6 \end{bmatrix}$

6. Suppose that an economy consists of three industries: a computing service, a statistical service, and an engineering service. For each \$1 of computing that is provided, 30¢ is spent on computing, 10¢ on statistical services and 30¢ on engineering. For each \$1 on statistical service, 20¢ is spent on computing, 40¢ on statistics, and 20¢ on engineering. Each \$1 in engineering takes 30¢ in computing, 10¢ in statistical services, and 30¢ in engineering. Suppose there is an external demand for \$1000 in computing, \$1500 in statistical services, and \$1800 in engineering.
   (a) Compute $\det(I - C)$.
   (b) Compute $(I - C)^{-1}$.
   (c) How much should each industry produce to meet the demand?

Linear Programming (The Simplex Method)

7. Solve using the simplex algorithm.
$$\begin{array}{lrcrcrl} \textbf{Maximize } z = & 3x_1 & - & 4x_2 & + & 5x_3 & \\ \text{subject to} & 2x_1 & - & 3x_2 & + & 5x_3 & \leq 30 \\ & -2x_1 & + & x_2 & + & x_3 & \leq 20 \\ & x_1 & + & 7x_2 & & & \leq 10 \\ & \multicolumn{6}{l}{x_1 \geq 0,\quad x_2 \geq 0,\quad x_3 \geq 0} \end{array}$$

8. Solve using the simplex algorithm.
$$\begin{array}{lrcrcrl} \textbf{Maximize } z = & 20x_1 & + & 5x_2 & - & 13x_3 & \\ \text{subject to} & x_1 & + & 4x_2 & - & x_3 & \leq 20 \\ & 2x_1 & + & 3x_2 & - & 2x_3 & \leq 30 \\ & -4x_1 & - & 9x_2 & + & 18x_3 & \leq 80 \\ & \multicolumn{6}{l}{x_1 \geq 0,\quad x_2 \geq 0,\quad x_3 \geq 0} \end{array}$$

9. Solve using the simplex algorithm.
$$\begin{array}{lrcrcrl} \textbf{Minimize } z = & -5x_1 & + & 4x_2 & - & 3x_3 & \\ \text{subject to} & x_1 & + & x_2 & - & 2x_3 & \leq 10 \\ & & & 7x_2 & + & x_3 & \leq 5 \\ & 5x_1 & - & 3x_2 & + & 2x_3 & \leq 15 \\ & \multicolumn{6}{l}{x_1 \geq 0,\quad x_2 \geq 0,\quad x_3 \geq 0} \end{array}$$

10. Solve using the simplex algorithm.
$$\begin{array}{lrcrl} \textbf{Maximize } z = & x_1 & + & x_2 & \\ \text{subject to} & x_1 & + & 3x_2 & \leq 15 \\ & 2x_1 & + & x_2 & \leq 10 \\ & x_1 & & & \leq 4 \\ & \multicolumn{4}{l}{x_1 \geq 0,\quad x_2 \geq 0} \end{array}$$

11. Solve using the simplex algorithm.

$$\begin{array}{llrcrcrcl}
\textbf{Minimize} & z = & -4x_1 & + & 3x_2 & - & 2x_3 & & \\
\text{subject to} & & 4x_1 & + & 5x_2 & & & \leq & 2 \\
& & 8x_1 & - & 2x_2 & + & 3x_3 & \leq & 8 \\
& & x_1 & \geq 0, & x_2 & \geq 0, & x_3 & \geq 0 &
\end{array}$$

12. Consider the optimization problem below.

$$\begin{array}{llrcrcrcl}
\textbf{Maximize} & z = & -2x_1 & + & 4x_2 & + & x_3 & & \\
\text{subject to} & & 5x_1 & + & x_2 & - & 3x_3 & \leq & 3 \\
& & 6x_1 & - & 2x_2 & + & 4x_3 & \leq & 7 \\
& & x_1 & \geq 0, & x_2 & \geq 0 & x_3 & \geq 0 &
\end{array}$$

(a) Use the simplex algorithm to show that the linear program is unbounded (has no maximum) by identifying an unbounded variable.

(b) Support your answer to part (a) by finding a feasible solution where $z \geq 1300$.

13. Consider the optimization problem below.

$$\begin{array}{llrcrcrcl}
\textbf{Minimize} & z = & 3x_1 & - & 6x_2 & + & 2x_3 & & \\
\text{subject to} & & x_1 & + & 2x_2 & - & 5x_3 & \leq & 3 \\
& & & & 2x_2 & - & x_3 & \leq & 2 \\
& & -5x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
& & x_1 & \geq 0, & x_2 & \geq 0, & x_3 & \geq 0 &
\end{array}$$

(a) Use the simplex algorithm to show that the linear program is unbounded (has no minimum) by identifying an unbounded variable.

(b) Support your answer to part (a) by finding a feasible solution where $z \leq -2012$

14. Consider the optimization problem below.

$$\begin{array}{llrcrcrcl}
\textbf{Minimize} & z = & -3x_1 & + & 4x_2 & + & 2x_3 & & \\
\text{subject to} & & -4x_1 & + & x_2 & + & x_3 & \leq & 3 \\
& & 2x_1 & - & 5x_2 & + & x_3 & \leq & 4 \\
& & -2x_1 & & & & x_3 & \leq & 1 \\
& & x_1 & \geq 0, & x_2 & \geq 0, & x_3 & \geq 0 &
\end{array}$$

(a) Use the simplex algorithm to show that the linear program is unbounded (has no minimum) by identifying an unbounded variable.

(b) Support your answer to part (a) by finding a feasible solution where $z \leq -7006$.

15. Consider the optimization problem below.

$$\begin{array}{llrcrcrcl}
\textbf{Maximize} & z = & -3x_1 & + & 4x_2 & + & 3x_3 & & \\
\text{subject to} & & -4x_1 & + & 2x_2 & + & x_3 & \leq & 8 \\
& & -2x_1 & + & x_2 & - & x_3 & \leq & 3 \\
& & x_1 & - & 3x_2 & & & \leq & 6 \\
& & x_1 & \geq 0, & x_2 & \geq 0, & x_3 & \geq 0 &
\end{array}$$

(a) Use the simplex algorithm to show that the linear program is unbounded (has no maximum) by identifying an unbounded variable.

(b) Support your answer to part (a) by finding a feasible solution where $z \geq 5012$.

16. You are selling martinis at an event to raise funds for charity. A dry martini is $5/3$ oz. of Gin and $1/3$ oz. of Vermouth. A medium martini is $3/2$ oz. Gin and $1/2$ oz. Vermouth. You have available 60 oz. of Gin and 16 oz. of Vermouth. You can sell dry martinis for \$8 each and medium martinis for \$10 each, but you only have 39 glasses for serving all of the drinks sold. How many of each kind of martini should be sold to maximize revenue? Name your variables, set up a linear program, and solve by using the simplex algorithm.

17. The Simple Machine Company makes Widgets, Gadgets, and Gizmos out of pulleys, wedges, and levers. Each Widget requires 3 pulleys, 2 wedges, and 4 levers. Each Gadget requires 7 pulleys and 1 lever. Each Gizmo requires 3 pulleys, 2 wedges, and 5 levers. The company has 56 pulleys, 16 wedges, and 33 levers available. Suppose the company sells Widgets for \$2 each, Gadgets for \$4 each, and Gizmos for \$1 each. Set-up and use the simplex algorithm to solve a linear program to determine how many Widgets, Gadgets, and Gizmos should be made to maximize revenue.

(a) What is the maximum revenue?

(b) When revenue is maximized, how many pulleys, wedges, and levers are left over?

18. The Furniture Factory makes beds, chairs, and couches from the raw materials labor, lumber, and cloth. The company makes a profit of \$60 per bed, \$10 per chair, and \$40 per couch. Each bed requires 1 hour of labor, 3 metres of lumber, and 3 metres of cloth; each chair requires 1 hour of labor, 5 meters of lumber and 7 meters of cloth; and each couch requires 2 hours of labor, 1 meter of lumber, and 1 meter of cloth. If there are 100 hours of labor, 90 meters of lumber, and 120 meters of cloth available, you would like to determine how many beds, chairs, and couches the company should make to maximize profit.

(a) Name variables and set up a linear program that represents this situation.

(b) What is the maximum profit?

(c) How many beds, chairs, and couches should be made to maximize profit?

(d) When profit is maximized, how much labor, lumber, and cloth will go unused?

19. Squeeky Cleaners makes three cleaning products: Ocean Fresh, Summer Breeze, and Lemon Zest, which sell for \$1, \$3, and \$2 respectiviely. The products are made using three processes: separating, blending, and mixing. One batch of Ocean Fresh requires 2 hours of separating, 1 hour of blending, and 4 hours of mixing. One batch of Summer Breeze requires 1 hour of separating and 2 hours of mixing. One batch of Lemon Zest requires 1 hour of blending and 1 hour of mixing. In a week it is possible to do 25 hours of separating, 30 hours of blending, and 40 hours of mixing. The company would like to determine how many batches of each product should be made to maximize weekly revenue.
    (a) Name variables and set-up a linear program that represents this situation.
    (b) Solve the problem using the simplex algorithm.
    (c) When revenue is maximized, how many hours of each process go unused?
20. Alice, Bob, and Cathy together make Xylophones, Yoyos, and Zippers. To make a Xylophone takes 4 hours of Alice's time, 2 hours of Bob's time, and 10 hours of Cathy's time. (So together it takes them 16 hours.) Similarly to make a Yoyo takes 9 hours for Alice, 3 hours for Bob, and 5 hours for Cathy; and to make a Zipper takes 3 hours, 1 hour, and 6 hours of their time respectively. Meanwhile, Alice has 120 hours available, Bob has 50 hours available, and Cathy has 400 hours available. Finally, Xylophones sell for \$12 each, Yoyos sell for \$8 each and Zippers sell for \$7 each.
    (a) Name variables and set-up a linear program to maximize revenue under these conditions.
    (b) Solve the program to determine how many Xylophones, Yoyos, and Zippers are made when revenue is maximized.
    (c) How many hours do Alice, Bob, and Cathy each have left over when revenue is maximized?
21. A popular cleaning brand produces two housecleaners: EnviroShine and Nature's Scent. Both cleaners contain the same basic ingredients, but in different proportions. To protect their trade secret, we will refer to the three ingredients as I_1, I_2, and I_3. These cleaners are sold in 1 L containers. The production of 1 L of EnviroShine requires 0.4 L of I_1, 0.3 L of I_2, and 0.3 L of I_3. The production of 1 L of Natrue's Scent requires 0.5 L of I_1, 0.2 L of I_2, and 0.3 L of I_3. The company's suppliers can guarantee a weekly supply of 94 L of I_1, 51 L of I_2, and 60 L of I_3. The profit margin is \$1.50 for every EnviroShine bottle, and \$1.20 for every Nature's Scent bottle.
    (a) How many litres of each housecleaner does the company need to produce weekly to maximize its profit?
    (b) What quantities of the 3 ingredients should the company purchase weekly to avoid needing to store surplus?
    (c) The distribution service informs the director of production that they are unable to sell all the EnviroShine stock. The reports for the last months show that the distributers cannot sell more than 70 bottles of this product per week. Given this infomation, establish a new weely production.
    (d) Does the revised work plan require that the weekly puchasing of ingredients be modified? If so, what are the new quantities of each ingredient that should be purchased weekly?

Markov Chains

22. In each of the following cases, find the state vector $\mathbf{x_3}$, given the transition matrix and state vector provided.
    (a) $P = \begin{bmatrix} 0.3 & 0.9 \\ 0.7 & 0.1 \end{bmatrix}$ and $\mathbf{x_0} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$
    (b) $P = \begin{bmatrix} 0.3 & 0.9 \\ 0.7 & 0.1 \end{bmatrix}$ and $\mathbf{x_1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
    (c) $P = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.6 & 0.8 & 0.3 \\ 0.1 & 0 & 0.2 \end{bmatrix}$ and $\mathbf{x_2} = \begin{bmatrix} 0.3 \\ 0.3 \\ 0.4 \end{bmatrix}$
23. For each of the following regular transition matrices, find the associated steady state vector.
    (a) $\begin{bmatrix} 0.3 & 0.9 \\ 0.7 & 0.1 \end{bmatrix}$
    (b) $\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$
    (c) $\begin{bmatrix} 0.3 & 0.1 & 0.5 \\ 0.4 & 0.8 & 0.1 \\ 0.3 & 0.1 & 0.4 \end{bmatrix}$

24. (a) Given $P = \begin{bmatrix} 1/3 & 1/2 \\ 2/3 & 1/2 \end{bmatrix}$, find P^3.

(b) Use your result from part (a) to answer the following question:

If my cat is awake at some moment, there is a $1/3$ chance that she will be awake an hour from then. If my cat is asleep at some moment, there is a $1/2$ chance that she will be awake an hour from then.

My cat is awake right now. What is the chance that she will be asleep three hours from now?

25. At a certain company, on any given day some employees are at work and the rest are absent. It is known that if an employee is at work today, there is a 80% chance that they will be at work tomorrow (and a 20% chance that they will be absent tomorrow), and if the employee is absent today, there is a 75% chance that they will be at work tomorrow (and a 25% chance that they will be absent tomorrow).

In the long term, what % of employees do we expect to be at work each day, and what % of employees do we expect to be absent?

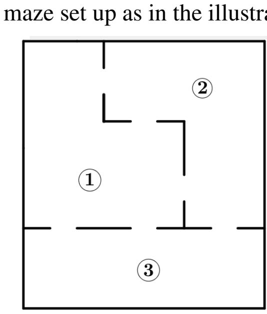
26. In a certain urban area the overall population remains stable, but every year 10% of the people who live in the urban core move out to the suburbs, and 25% of the people who live in the suburbs move in to the urban core.

In the long term, what % of people do we expect to live in the urban core, and what % do we expect to live in the suburbs?

27. At a certain college, on any given day some students buy lunch in the cafeteria and the rest bring a lunch from home. It is known that if a student eats in the cafeteria today, there is a 34% chance that they will eat in the cafeteria tomorrow, and if they bring a lunch from home today, there is a 44% chance that they will bring a lunch from home tomorrow.

On a day in the distant future, what % of students do we expect to be buying their lunch in the cafeteria, and what % of students will be bringing a lunch from home?

28. A rat is placed in a maze set up as in the illustration below:



When changing chambers (numbered as chambers 1, 2, and 3 in the image), the rat will choose a doorway at random from among those it can see. For example, if the rat is in Chamber 1, it can see five doorways, three of which will lead into Chamber 2, and two of which leads into Chamber 3. That means that there is a $3/5$ probability that the next Chamber that the rat is to occupy will be Chamber 2, and a $2/5$ probability that it will occupy Chamber 3 next.

(a) Determine the transition matrix P associated with a rat's movement from one chamber to another.

(b) If a rat is currently in Chamber 2, what is the probability that it will be in Chamber 1, Chamber 2, and Chamber 3, respectively, after changing chamber twice?

(c) What is the long-term probability that the rat will be in Chamber 1?

29. History has shown that a student who asks for an extension on a homework assignment in Linear Algebra is 40% likely to also need an extension on the next homework assignment. Meanwhile, a student who does not ask for an extension on their current homework assignment is only 10% likely to ask for an extension on their next homework assignment. If there is a 75% chance that Sophie will be able to complete Assignment 4 on time, what is the probability that she will need an extension on Assignment 6?

30. There is a 45% probability that any given child will grow up to become a doctor if their father is a doctor. Meanwhile, a child whose father is *not* a doctor is only 5% likely to become a doctor when they grow up. Dr. Peter Cummings has many sons, but no daughters. What is the probability that his first grandchild will grow up to become a doctor?

31. Although it isn't true that all cats have nine lives, something strange has been happening to felines across Canada. A virus swept through the feline population in January 2015 and, since then, the rules of life and death haven't applied as reliably as in previous years. On any given month after January 2015, a living cat will have a 90% chance of also being alive the next month. However, since the mysterious virus hit, it has been noted that a cat that is dead on one month will have a 98% chance of remaining dead next month, but also a 2% chance of spontaneously coming back to life (assuming that the cat died after the virus had struck).

(a) Determine the transition matrix P associated with this situation.

(b) If Mr. Whiskers (your neighbour's cat) is alive today, what is the probability that Mr. Whiskers will be dead two months from now?

(c) Assuming that no new cats are born, what percentage of the cat population from January 2015 is expected to be alive, and what percentage of the cat population is expected to be dead in the distant future?

32. Working with $\bmod(26)$ calculations:
    (a) Explain why 7 is considered to be the multiplicative inverse of 15 in the cotext of $\bmod(26)$ calculations.
    (b) Use matrix multiplication to determine which of the following matrices (B or C) is an inverse matrix $\bmod(26)$ of the matrix $A = \begin{bmatrix} 5 & 9 \\ 1 & 4 \end{bmatrix}$. (In other words, which of the two matrices below would operate as a decryption matrix for the encoding matrix A?)
    $$B = \begin{bmatrix} 24 & 11 \\ 7 & 17 \end{bmatrix} \qquad C = \begin{bmatrix} 4 & 3 \\ 9 & 12 \end{bmatrix}$$
33. Encode the word **JELLY** using a Hill 2-cipher with the encoding matrix $A = \begin{bmatrix} 2 & 7 \\ 3 & 10 \end{bmatrix}$.
34. Below are four encoding matrices. In each case, find the decryption matrix.
    (a) $A = \begin{bmatrix} 3 & 2 \\ 4 & 11 \end{bmatrix}$
    (b) $A = \begin{bmatrix} 8 & 5 \\ 5 & 5 \end{bmatrix}$
    (c) $A = \begin{bmatrix} 7 & 3 \\ 4 & 3 \end{bmatrix}$
    (d) $A = \begin{bmatrix} 9 & 2 \\ 5 & 3 \end{bmatrix}$
35. Using the encryption matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix}$, decode the ciphertext below to reveal a request:

    XSQXXRAJYPMC
36. Using the encryption matrix $A = \begin{bmatrix} 7 & 5 \\ 2 & 15 \end{bmatrix}$, decode the ciphertext below to answer the riddle, "What's the difference between 0 and 8?"

    QFQHFB
37. Using the encryption matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$, decode the ciphertext below to reveal the plea:

    WOHZBY
38. Using the encryption matrix $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$, decode the ciphertext below to reveal the title of a great movie:

    REREHXDW
39. Using the encryption matrix $A = \begin{bmatrix} 1 & 3 \\ 5 & 6 \end{bmatrix}$, decode the ciphertext below to reveal a scientific feat:

    RRBSOIMJBLWRIUBW
40. Answers to the following trivia questions are provided in encypted form, each time using one of the three following encryption matrices:
    $$A = \begin{bmatrix} 4 & 9 \\ 5 & 12 \end{bmatrix} \qquad B = \begin{bmatrix} 6 & 5 \\ 5 & 7 \end{bmatrix} \qquad C = \begin{bmatrix} 5 & 4 \\ 1 & 3 \end{bmatrix}$$
    Decrypt the answers to each of the questions below using the encryption matrices mentioned in each case.
    (a) What is the longest river in Europe?
    ODGNMQ, encrypted using A
    (b) Who wrote *The Catcher in the Rye*?
    NVUVRMTISG, encrypted using C
    (c) What word describes a dozen dozen?
    HQWQMK, encrypted using A
    (d) Who won the Nobel Prize in Litterature in 2016?
    IKFLBAXY, encrypted using B
    (e) In 2016, where was is the world's tallest freestanding stucture located?
    YKQQUD, encrypted using B
    (f) Who painted *Luncheon of the Boating Party*?
    FGZGMK, encrypted using C
    (g) In physics, what is defined as the measure of a rotational force on an object?
    GTQHYI, encrypted using A
    (h) What is the average gestation period (in months) for an African elephant?
    AAVSKOAAIX, encrypted using B
41. A spy arrives in a foreign country and, in a train station locker, finds and opens a message that reads

    *Go to **UXGHDZHKWY** and introduce yourself as **IDHKXEBL**. You contact will meet you there and identify himself by using the phrase **QQZLWDUMQKGI** into his first sentence. He will give you further instructions.*

    Before leaving your previous contact, you were asked to memorize the following three encoding matrices:
    $$A_1 = \begin{bmatrix} 2 & 9 \\ 1 & 8 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 10 & 1 \\ 11 & 3 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 4 & 1 \\ 7 & 7 \end{bmatrix}$$
    Use the encoding matrices, in order, to decode the message from the locker.
    (a) What three decryption matrices will you use?
    (b) Where must you go?
    (c) What name should you use to introduce yourself?
    (d) What phrase will your contact use to identify himself?

42. You have a hard time memorizing dates and names and, during a test on 19^{th} century America, you can't seem to remember who was President of the Unites States in 1867. Luckily, you're quite good at math, and so is your best friend, who tosses you her answer in the following note, confident that your history teacher doesn't understand the Hill 2-cipher and would ignore it:

$$\textbf{UTIBTODT} \qquad \textit{Use} \qquad A = \begin{bmatrix} 8 & 9 \\ 3 & 4 \end{bmatrix}$$

Unfortunately, your best friend is also pretty bad at remembering names and dates, and she forgets that the correct answer is Andrew Johnson. What (incorrect) answer was in the note?

43. Sally and Steve have been married for ten years and love to leave each other notes written in the form of a Hill 2-cipher. This morning, Sally left the following note for Steve:

DCUHGORKKHWVXUBC

$$\textit{Encoding matrix } A = \begin{bmatrix} 3 & 7 \\ 3 & 12 \end{bmatrix}$$

What was the message?

ANSWERS:

1. (a) \$160,000 of energy and \$70,000 of material should be produced.
   (b) Yes, the economy is productive since $(I-C)^{-1} \geq 0$.
   (c) The economy consumes \$158,000 of energy and \$69,000 of material.
2. (a) \$19,000 of Iron and \$10,000 of Steel should be produced.
   (b) Only iron is profitable.
3. (a) The economy should produce \$39000 in goods and \$31000 in services.
   (b) Yes, the economy is productive since $(I-C)^{-1} \geq 0$.
4. (a) \$2100 in services and \$2600 in manufacturing should be produced.
   (b) \$1200 in services and \$1100 in manufacturing is consumed internally.
   (c) Both industries are profitable.
5. (a) Both industries are profitable, and the economy is productive.
   (b) Only the second industry is profitable, and the economy is productive.
   (c) Only the second industry is profitable, and the economy is NOT productive.
6. (a) $\det(I-C) = .2$
   (b) $(I-C)^{-1} = \dfrac{1}{.2}\begin{bmatrix} .4 & .2 & .2 \\ .1 & .4 & .1 \\ .2 & .2 & .4 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ .5 & 2 & .5 \\ 1 & 1 & 2 \end{bmatrix}$
   (c) \$5300 in computing, \$4400 in statistical services and \$6100 in engineering should be produced.
7. Max $z = 40$ at $(10, 0, 2, 0, 38, 0)$.
8. Max $z = 370$ at $(25, 0, 10, 5, 0, 0)$
9. Min $z = -20$ at $(1, 0, 5, 19, 0, 0)$.
10. Max $z = 7$ at $(3, 4, 0, 0, 1)$.
11. Min $z = -16/3$ at $(0, 0, 8/3, 2, 0)$.
12. (a) x_3 is unbounded
    (b) example: $z = 1312$ at $(0, 303, 100, 0, 213)$
13. (a) x_1 is unbounded
    (b) example: $z = -2012$ at $(1000, 2504, 5006, 19025, 0, 0)$
14. (a) x_2 is unbounded
    (b) example: $z = -7006$ at $(5002, 2000, 0, 18011, 0, 10005)$
15. (a) x_1 is unbounded
    (b) example: $z = 5012$ at $(1000, 0, 2003, 2, 0, 5015)$
16. Let $x =$# of dry martinis, $y =$# of medium martinis.
$$\begin{array}{llrcrcl} \textbf{Maximize} & z = & 8x & + & 10y & & \\ \text{subject to} & & \frac{5}{3}x & + & \frac{3}{2}y & \leq & 60 \\ & & \frac{1}{3}x & + & \frac{1}{2}y & \leq & 16 \\ & & x \geq 0, & & y \geq 0 & & \end{array}$$
    Maximum Revenue is \$344 when 18 dry martinis and 20 medium martinis are sold ($s_1 = 0, s_2 = 0, s_3 = 1$).
17. Let x_1 =number of Widgets made; let x_2 =number of Gadgets made; and let x_3=number of Gizmos made.
$$\begin{array}{llrcrcrcl} \textbf{Maximize} & z = & 2x_1 & + & 4x_2 & + & x_3 & & \\ \text{subject to} & & 3x_1 & + & 7x_2 & + & 3x_3 & \leq & 56 \\ & & 2x_1 & & & + & 2x_3 & \leq & 16 \\ & & 4x_1 & + & x_2 & + & 5x_3 & \leq & 33 \\ & & x_1 \geq 0, & & x_2 \geq 0, & & x_3 \geq 0 & & \end{array}$$
    The maximum revenue is \$34 when 7 widgets, 5 gadgets, and 0 gizmos are made. When revenue is maximized there will be 0 pulleys, 2 wedges, and 0 levers remaining.

18. (a) Let x_1 be the number of beds, x_2 be the number of chairs, and x_3 be the number of couches made.

$$\begin{array}{lrcrcrl} \textbf{Maximize } z= & 60x_1 & + & 10x_2 & + & 40x_3 & \\ \text{subject to} & x_1 & + & x_2 & + & 2x_3 & \leq 100 \\ & 3x_1 & + & 5x_2 & + & x_3 & \leq 90 \\ & 3x_1 & + & 7x_2 & + & x_3 & \leq 120 \\ & x_1 \geq 0, & & x_2 \geq 0, & & x_3 \geq 0 & \end{array}$$

(b) Max profit = $2640.

(c) Profit is maximized when 16 beds, 0 chairs, and 42 couches are made.

(d) When profit is maximized, there are 0 hours of labor, 0 meters of lumber, and 30 meters of cloth unused.

19. (a) Let x_1 =number of batches of Ocean Fresh; let x_2 =number of batches of Summer Breeze; and let x_3=number of batches of Lemon Zest.

$$\begin{array}{lcrcrcrcl} \textbf{Maximize} & z = & x_1 & + & 3x_2 & + & 2x_3 & & \\ \text{subject to} & & 2x_1 & + & x_2 & & & \leq & 25 \\ & & x_1 & & & + & x_3 & \leq & 30 \\ & & 4x_1 & + & 2x_2 & + & x_3 & \leq & 40 \\ & & x_1 \geq 0, & & x_2 \geq 0, & & x_3 \geq 0 & & \end{array}$$

(b) The maximum revenue is $75 when 0 batches of Ocean Fresh, 5 batches of Summer Breeze, and 30 batches of Lemon Zest are made.

(c) When revenue is maximized there will be 20 hours of separating, 0 hours of blending, and 0 hours of mixing remaining.

20. (a) Let x_1 =# of Xylophones; let x_2 =# of Yoyos; and let x_3 =# of Zippers.

$$\begin{array}{lcrcrcrcl} \textbf{Maximize} & z = & 12x_1 & + & 8x_2 & + & 7x_3 & & \\ \text{subject to} & & 4x_1 & + & 9x_2 & + & 3x_3 & \leq & 120 \\ & & 2x_1 & + & 3x_2 & + & x_3 & \leq & 50 \\ & & 10x_1 & + & 5x_2 & + & 6x_3 & \leq & 400 \\ & & x_1 \geq 0, & & x_2 \geq 0, & & x_3 \geq 0 & & \end{array}$$

(b) Maximum revenue of $320 occurs when 15 Xylophones, 0 Yoyos, and 20 Zippers are made.

(c) When revenue is maximized Alice and Bob have 0 hours left over, while Cathy has 130 hours left over.

21. (a) Let x be the number of litres of EnviroShine, and let y be the number of litres of Nature's Scent cleaners.

$$\begin{array}{lcrcrcl} \textbf{Maximize} & z = & 1.5x & + & 1.2y & & \\ \text{subject to} & & 0.4x & + & 0.5y & \leq & 94 \\ & & 0.3x & + & 0.2y & \leq & 51 \\ & & 0.3x & + & 0.3y & \leq & 60 \\ & & x \geq 0, & & y \geq 0 & & \end{array}$$

Maximum profit = $273 at (110,90,5,0,0).

Other corner points: $(0,0), (0,188), (60,140), (170,0)$.

(b) 89 L of I_1, 51 L of I_2 and 60 L of I_3.

(c) New constraint: $x \leq 70$. New production: $(70, 130, 1, 4, 0, 0)$ with a profit of $261.

(d) The weekly purchasing must be modified to 93 L of I_1, 47 L of I_2 and 60 L of I_3.

22. (a) $\mathbf{x_3} = \begin{bmatrix} 72/125 \\ 53/125 \end{bmatrix}$

(b) $\mathbf{x_3} = \begin{bmatrix} 9/25 \\ 16/25 \end{bmatrix}$

(c) $\mathbf{x_3} = \begin{bmatrix} 7/20 \\ 27/50 \\ 11/100 \end{bmatrix}$

23. Steady state vectors:

(a) $\begin{bmatrix} 9/16 \\ 7/16 \end{bmatrix}$

(b) $\begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix}$

(c) $\begin{bmatrix} 11/48 \\ 9/16 \\ 5/24 \end{bmatrix}$

24. (a) $\begin{bmatrix} 23/54 & 31/72 \\ 31/54 & 41/72 \end{bmatrix}$

(b) There's a $31/54$ chance (or about 57%) that my cat will be asleep three hours from now.

25. Approximately 79% at work and 21% absent.

26. Approximately 71% in the urban core and 29% in the suburbs.

27. Approximately 46% will be buying lunch in the cafeteria, and 54% will be bringing a lunch from home.

28. (a) $P = \begin{bmatrix} 0 & 3/4 & 2/3 \\ 3/5 & 0 & 1/3 \\ 2/5 & 1/4 & 0 \end{bmatrix}$

(b) There is a $1/6$ probability that the rat will be in Chamber 1, a $8/15$ probability that it will be in Chamber 2, and a $3/10$ probability that it will be in Chamber 3.

(c) There is a $5/12$ probability that the rat will be in Chamber 1 long-term. (Note that this means that, over the course of an extended period of time, the rat will wnd up in Chamber 1 roughly $5/12$ of the time.)

29. There is a $61/400$ (15.25%) chance that Sophie will need an extension on Assignment 6.

30. There is a 23% chance that Dr. Cummings' first grandchild will grow up to become a doctor.

31. (a) $P = \begin{bmatrix} 0.9 & 0.02 \\ 0.1 & 0.98 \end{bmatrix}$

(b) There is a 18.8% chance that Mr. Whiskers will be dead two months from now.

(c) At any given point in the distant future, $1/6$ of the cat population from January 2015 would be alive, and $5/6$ would be dead.

32. (a) 7 and 15 are multiplicative inverses mod(26) because $(7)(15) = 105 \equiv 1 \bmod(26)$.

(b) B is the inverse of A (and C is not) because

$$AB = \begin{bmatrix} 183 & 208 \\ 52 & 131 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{mod}(26)$$
$$BA = \begin{bmatrix} 131 & 260 \\ 52 & 131 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{mod}(26)$$
$$\text{but } AC = \begin{bmatrix} 121 & 123 \\ 40 & 51 \end{bmatrix} \equiv \begin{bmatrix} 17 & 19 \\ 14 & 25 \end{bmatrix} \text{mod}(26)$$

33. **CBDZQM**

34. (a) $A^{-1} = \begin{bmatrix} 15 & 2 \\ 4 & 23 \end{bmatrix}$

    (b) $A^{-1} = \begin{bmatrix} 9 & 17 \\ 17 & 4 \end{bmatrix}$

    (c) $A^{-1} = \begin{bmatrix} 9 & 17 \\ 14 & 21 \end{bmatrix}$

    (d) $A^{-1} = \begin{bmatrix} 17 & 6 \\ 15 & 25 \end{bmatrix}$

35. PASS THE SUGAR

36. A BELT

37. HELP ME

38. MEMENTO

39. THEY SPLIT THE ATOM

40. Decryption matrices: $A^{-1} = \begin{bmatrix} 4 & 23 \\ 7 & 10 \end{bmatrix}$

    $B^{-1} = \begin{bmatrix} 5 & 15 \\ 15 & 8 \end{bmatrix}$ $\quad C^{-1} = \begin{bmatrix} 5 & 2 \\ 7 & 17 \end{bmatrix}$

    (a) VOLGA

    (b) J.D. SALINGER

    (c) GROSS

    (d) BOB DYLAN

    (e) RENOIR

    (f) DUBAI

    (g) TORQUE

    (h) TWENTY-TWO

41. (a) $A_1^{-1} = \begin{bmatrix} 16 & 21 \\ 11 & 4 \end{bmatrix}$ $\quad A_2^{-1} = \begin{bmatrix} 7 & 15 \\ 9 & 6 \end{bmatrix}$

    $A_3^{-1} = \begin{bmatrix} 9 & 21 \\ 17 & 20 \end{bmatrix}$

    (b) HOTEL RUBIO

    (c) SAM HILL

    (d) PERFECT STORM

42. LINCOLN (using $A^{-1} = \begin{bmatrix} 6 & 19 \\ 15 & 12 \end{bmatrix}$)

43. GET MILK. I LOVE YOU. (using $A^{-1} = \begin{bmatrix} 6 & 3 \\ 5 & 21 \end{bmatrix}$)