# DDC Final Exam
**May 12th, 2017**

1. **[2]** Write $z = -2\sqrt{3} + 2i$ in polar form.

2. **[2]** Find the real and imaginary parts of $\frac{1-2i}{3+4i}$.

3. **[2]** Find all complex solutions to $z^3 = -8$.

4. **[6]** In $\mathbb{P}_2$:

   (a) Find the change-of-coordinates matrix from the basis $\mathcal{B} = \{1+t^2, 1-t, -t+t^2\}$ to the standard basis $\mathcal{C} = \{1, t, t^2\}$.

   (b) Given that $[p]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, find $[p]_{\mathcal{C}}$ and $p$.

   (c) Given $q = 1 + 2t + 3t^2$, how would you use matrix multiplication to find $[q]_{\mathcal{B}}$? Just briefly explain what you would do without actually doing it.

5. **[5]** Find the eigenvalues of $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

6. **[5]** Let $A = \begin{bmatrix} 3 & -5 \\ 1 & -1 \end{bmatrix}$. Construct the general solution of $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$ involving complex eigenfunctions.

7. **[5]** Find the least-squares line $y = ax + b$ for the data points $(0,1), (1,3), (2,2), (3,3)$.

8. **[6]** Let $A = \begin{bmatrix} 2 & -2 & 18 \\ 2 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$.

   (a) Find an orthonormal basis for $\mathrm{Col}(A)$.

   (b) Find a QR factorization of $A$.

9. **[7]** Consider $\mathbb{P}_2$ together with the inner product
   $(p,q) = p(0)q(0) + p(1)q(1) + p(2)q(2)$.

   (a) Compute $\|3 - 2t\|$.

   (b) Find the **orthogonal** projection of $t^2$ onto the subspace spanned by 1 and $t$.

10. **[6]** Find the second order Fourier approximation to the function
$$f(x) = \begin{cases} 0 & 0 \le x < \pi \\ 1 & \pi \le x \le 2\pi \end{cases}$$
on the interval $[0, 2\pi]$.

11. **[7]** Consider the quadratic form on $\mathbb{R}^2$:
$$Q(\mathbf{x}) = 16x_1^2 - 8x_1x_2 + x_2^2.$$

   (a) Find a symmetric matrix $A$ so that $Q(x) = x^T A x$.

   (b) Make a change of variable $x = Py$, $P$ orthogonal, that transforms $Q$ into a quadratic form with no cross-product term. Clearly state $P$ and the new quadratic form.

   (c) Classify $Q$.

12. **[5]** Find the singular values of $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$.

13. **[5]** A singular value decomposition of $A$ is $A = U\Sigma V^T$ where
$$U = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix},$$
$$V^T = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{18} & -2/3 \\ 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \\ 0 & -4/\sqrt{18} & 1/3 \end{bmatrix}^T$$

   (a) Find a **reduced** singular value decomposition of $A$.

   (b) Find a **reduced** singular value **expansion** of $A$.

14. **[5]** Find the minimum polynomial of the given matrix.

   (a) $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$

15. **[6]** In this question $H_1$, $H_2$, and $H_3$ are subspaces of $\mathbb{R}^3$. Show that the sum is direct, or explain why it is not.

   (a) $H_1 + H_2 + H_3$ where $H_1 = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$,
$$H_2 = \mathrm{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}, \quad H_3 = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

   (b) $H_1 + H_2 + H_3$ where $H_1 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y + z = 0 \right\}$
$$H_2 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x - y - z = 0 \right\}, \quad H_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

16. **[6]** Find the Jordan canonical form of
$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \quad \text{Hint: } m_A(\lambda) = \lambda^2(\lambda - 2).$$

17. **[2]** Let $z \in \mathbb{C}$. If $|z| = Re(z)$ prove that:
   (a) $z \in \mathbb{R}$; (b) $z \ge 0$.

18. **[3]** The matrix $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 1 & 1 \end{bmatrix}$ is the change of basis matrix from what basis $\mathcal{B}$ to the basis

$$\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \quad \text{for } \mathbb{R}^3?$$

19. **[4]** Let $A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$. Use diagonalization to find a formula for $A^k$ where $k$ is a positive integer.

20. **[2]** Let $A$ be $n \times n$ and let $\lambda_1$ and $\lambda_2$ be distinct eigenvalues of $A$. Show that the intersection of the corresponding eigenspaces is $\{\mathbf{0}\}$.

21. **[3]** For which value(s) of $a \in \mathbb{R}$ is the matrix $A = \begin{bmatrix} 1 & 3 & a \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ diagonalizable?

22. **[3]** Let $u$ and $v$ be vectors in an inner product space, and let $H = \text{Span}\{u, v\}$. Show that $w \in H^{\perp}$ **if and only if** $\langle w, u \rangle = \langle w, v \rangle = 0$.

23. **[3]** Assume that $A$ is a square matrix with minimum polynomial $m_A(\lambda) = \lambda(\lambda - 2)(\lambda + 3)$.
    (a) Explain why $A^2 + A - 6I \neq 0$.
    (b) Express $A^3$ as a linear combination of $A^2$ and $A$.

## ANSWERS

1. $z = 4e^{5\pi i/6}$

2. $-\frac{1}{5}, -\frac{2}{5}$

3. $z = 2e^{i\phi}$, $\phi = \pi/3, \pi, 5\pi/3$

4. (a) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$
   (b) $[p]_{\mathcal{C}} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$, $p = 2 - 2t + 2t^2$
   (c) Calculate $P^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ where $P$ is the matrix found in (a).

5. $\lambda = 0, 1$

6. $\mathbf{x} = c_1 e^{(1+i)t} \begin{bmatrix} 2+i \\ 1 \end{bmatrix} + c_2 e^{(1-i)t} \begin{bmatrix} 2-i \\ 1 \end{bmatrix}$

7. $y = \frac{1}{2}x + \frac{3}{2}$

8. (a) $\left\{ \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}, \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix} \right\}$
   (b) $\begin{bmatrix} 2/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & -2/3 \\ 1/3 & 2/3 & 2/3 \end{bmatrix} \begin{bmatrix} 3 & 0 & 12 \\ 0 & 3 & -12 \\ 0 & 0 & 6 \end{bmatrix}$

9. (a) $\sqrt{11}$
   (b) $\frac{5}{3} + 2(t-1)$

10. $\frac{1}{2} - \frac{2}{\pi}\sin x$

11. (a) $\begin{bmatrix} 16 & -4 \\ -4 & 1 \end{bmatrix}$
    (b) $P = \frac{1}{\sqrt{17}} \begin{bmatrix} -4 & 1 \\ 1 & 4 \end{bmatrix}$, $Q' = 17y_1^2$
    (c) Positive semi-definite

12. $\sigma = \sqrt{3}, 1$

13. (a) $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{18} \\ 1/\sqrt{2} & 1/\sqrt{18} \\ 0 & -4/\sqrt{18} \end{bmatrix}^T$
    (b) $5 \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$
    $+ 3 \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1/\sqrt{18} & 1/\sqrt{18} & -4/\sqrt{18} \end{bmatrix}$

14. (a) $(\lambda - 2)^2$
    (b) $\lambda$
    (c) $\lambda - 3$

15. (a) Sum is direct. One should check that all the appropriate intersections are trivial.
    (b) Not direct: $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \in H_1 \cap H_2$

16. $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ -1/2 & 1/2 & 0 \end{bmatrix}$

17. (a) $|z| = Re(z) \Rightarrow \sqrt{a^2 + b^2} = a \Rightarrow b = 0 \Rightarrow z = a \in \mathbb{R}$
    (b) $z = a = \sqrt{a^2 + b^2} \geq 0$

18. $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \right\}$

19. $\begin{bmatrix} 2^k & 0 \\ 3^k - 2^k & 3^k \end{bmatrix}$

20. $Ax = \lambda_1 x$ and $Ax = \lambda_2 x \Rightarrow \lambda_1 x = \lambda_2 x \Rightarrow (\lambda_1 - \lambda_2)x = \mathbf{0} \Rightarrow x = \mathbf{0}$ (since $\lambda_1 \neq \lambda_2$)

21. $a = -6$

22. $(\Rightarrow)$: $w \in H^{\perp} \Rightarrow \langle w, x \rangle = 0\ \forall x \in H \Rightarrow \langle w, u \rangle = \langle w, v \rangle = 0$
    $(\Leftarrow)$: $\langle w, u \rangle = \langle w, v \rangle = 0 \Rightarrow \forall a, b \in \mathbb{R}$ we have $\langle w, au + bv \rangle = a\langle w, u \rangle + b\langle w, v \rangle = 0 + 0 = 0 \Rightarrow w \in H^{\perp}$

23. (a) The minimum polynomial of $A$ is of degree 3, so no polynomial $p$ of degree 2 can satisfy $p(A) = 0$.
    (b) $m_A(A) = 0 \Rightarrow A(A - 2I)(A + 3I) = 0 \Rightarrow A^3 + A^2 - 6A = 0 \Rightarrow A^3 = -A^2 + 6A$