DDC Final Exam (May 2018)

1. **[1]** Write $14e^{i\frac{3\pi}{2}}$ in rectangular form.
2. **[1]** Find i^{-63}. Write your answer in rectangular form.
3. **[2]** Find the real and imaginary parts of $\frac{4}{5-3i}$.
4. **[2]** Sketch the set $\{z \in \mathbb{C} : |z-4-3i| = 3\}$ in the complex plane.
5. **[6]** Given $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$, $\mathcal{E} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$, $\mathbf{x} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$, $[\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$ find:

   (a) $[\mathbf{x}]_{\mathcal{B}}$ (b) $[\mathbf{x}]_{\mathcal{E}}$ (c) $\mathbf{y}$ (d) $[\mathbf{y}]_{\mathcal{E}}$

6. **[5]** Given that the eigenvalues of $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$ are $\lambda = 1, 2$, diagonalize A (find P and D such that $A = PDP^{-1}$).
7. **[4]** Define $T : \mathbb{P}_1 \to \mathbb{R}^2$ by $T(p) = \begin{bmatrix} p(2) \\ p(3) \end{bmatrix}$.

   (a) Find the matrix of T relative to the basis $\{1, t\}$ for $\mathbb{P}_1$ and the standard basis for $\mathbb{R}^2$.

   (b) Find the matrix of T relative to the basis $\mathcal{B} = \{t, 1-2t\}$ for $\mathbb{P}_1$ and the basis $\mathcal{C} = \left\{ \begin{bmatrix} 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right\}$ for $\mathbb{R}^2$.

8. **[5]** Let $A = \begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}$.

   (a) Find the eigenvalues of A.

   (b) Factor A into SR where S is a scaling and R is a rotation. Be sure to identify the angle of rotation.

9. **[5]** Let $\mathbf{x}_k = \begin{bmatrix} O_k \\ R_k \end{bmatrix}$ where O_k is the owl population and R_k is the wood rat population in a certain region at time k. Assume further that $\mathbf{x}_{k+1} = A\mathbf{x}_k$ where $A = \begin{bmatrix} 0.1 & 0.6 \\ -0.3 & 1 \end{bmatrix}$.

   (a) Find a formula for $\mathbf{x}_k$, given that $\mathbf{x}_0 = \begin{bmatrix} 110 \\ 70 \end{bmatrix}$.

   (b) Does the wood rat population grow or decline?

10. **[5]** Write y as the sum of a vector in $W = \text{Span}\{u_1, u_2\}$ and a vector in $W^\perp$. Use the dot product.

    (a) $y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

    (b) $y = \begin{bmatrix} 4 \\ -6 \\ 1 \end{bmatrix}$, $u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

11. **[5]** Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

    (a) Find an orthogonal basis for $\text{Col}(A)$.

    (b) Find a QR factorization of A.

12. **[6]** Let $\mathbb{P}_2$ (polynomials of degree at most two) have the inner product $\langle p, q \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$.

    (a) Use the Gram-Schmidt process to find an orthogonal basis for $H = \text{Span}\{1,\ t^2\}$.

    (b) Find an **orthonormal** basis for $H^\perp$.

13. **[6]** Find the second order Fourier approximation of $f(x) = 1 + x$ over the interval $[0, 2\pi]$.
14. **[6]** Consider the quadratic form on $\mathbb{R}^2$: $Q(\mathbf{x}) = -6x_1x_2$

    (a) Find a symmetric matrix A such that $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$.

    (b) Make a change of variable $\mathbf{x} = P\mathbf{y}$, P orthogonal, that transforms Q into a quadratic form with no cross-product term. Clearly state P and the new quadratic form.

    (c) Classify Q.

15. **[6]** Find a singular value decomposition of $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$.

    Hint: The eigenvalues of $A^T A$ are 18 and 0.

16. **[5]** Find the minimum polynomial of $A = \begin{bmatrix} -2 & -1 & 1 \\ 2 & 1 & -3 \\ 0 & 0 & -1 \end{bmatrix}$

    Hint: the eigenvalues are 0 and -1.

17. **[6]** Given $H = \left\{ \begin{bmatrix} x \\ x \\ x \end{bmatrix} : x \in \mathbb{R} \right\}$, find:

    (a) Subspaces H_1, H_2 of $\mathbb{R}^3$, so that $H \oplus H_1 \oplus H_2 = \mathbb{R}^3$

    (b) H_3, a subspace of $\mathbb{R}^3$, such that $H^\perp \oplus H_3 = \mathbb{R}^3$

    (c) H_4, a subspace of $\mathbb{R}^3$, such that
    $$H \oplus H_4 = \left\{ \begin{bmatrix} 2x \\ x+y \\ 2y \end{bmatrix} : x, y \in \mathbb{R} \right\}$$

18. **[5]** Find the Jordan canonical form of $A = \begin{bmatrix} -2 & 2 & 1 \\ -7 & 4 & 2 \\ 6 & 0 & 0 \end{bmatrix}$. Hint: $m_A(\lambda) = \lambda^2(\lambda - 2)$.
19. **[2]** For which value(s) of $a \in \mathbb{R}$ is $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & 4 \end{bmatrix}$ diagonalizable?
20. **[3]** Let $V = \text{Span}\{f_1, f_2\}$ where $f_1 = 4\sin x - \cos x$, $f_2 = -2\sin x + 2\cos x$, $g_1 = 2\sin x + \cos x$, $g_2 = 3\cos x$. Find the transition matrix from $\mathcal{B} = \{f_1, f_2\}$ to $\mathcal{C} = \{g_1, g_2\}$.
21. **[4]** Let $A = \begin{bmatrix} a & b \\ a & b \end{bmatrix}$ where $a, b \in \mathbb{R}$ and a and b are not both zero.

(a) Find the eigenvalues of A.

(b) Describe the eigenspaces of A.

(c) Find an a and b that make A **not** diagonalizable.

22. **[2]** Let A be a 2×2 real symmetric matrix with two distinct eigenvalues. If $\begin{bmatrix} b \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ 4 \end{bmatrix}$ are eigenvectors of A, what are the possible values of b?

23. **[3]** Let $A = \begin{bmatrix} 3/5 & c & 4/5 \\ a & 1 & e \\ b & d & f \end{bmatrix}$ be orthogonal where $a, b, c, d, e, f \in \mathbb{R}$ and $b > 0$. Find a, b, c, d, e, and f.

24. **[2]**

    (a) Find 2×2 matrices A and B, $A \neq B$, that have the same characteristic polynomial and the same minimum polynomial.

    (b) Find 2×2 matrices A and B that have the same characteristic polynomial but different minimum polynomials.

25. **[3]** Let $z, w \in \mathbb{C}$ such that $\bar{z}w \neq 1$ and either $|z| = 1$ or $|w| = 1$. Show that $\left| \dfrac{z - w}{1 - \bar{z}w} \right| = 1$.

Answers

1. $-14i$

2. i

3. $\frac{10}{17}, \frac{6}{17}$

4. Circle of radius 3 in the complex plane, centered at $4 + 3i$.

5. (a) $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} 5 \\ -4 \end{bmatrix}$ (c) $\begin{bmatrix} 11 \\ -10 \end{bmatrix}$ (d) $\begin{bmatrix} 11 \\ -10 \end{bmatrix}$

6. $P = \begin{bmatrix} -2 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

7. (a) $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$

   (b) $\begin{bmatrix} 1/2 & 0 \\ 0 & -1 \end{bmatrix}$

8. (a) $1 \pm i\sqrt{3}$

   (b) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}$, $\quad \theta = -\pi/3$

9. (a) $\mathbf{x}_k = 30(0.7)^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 40(0.4)^k \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

   (b) $0 < \lambda < 1$ (*both*) $\implies$ decline.

10. (a) $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$

    (b) $\begin{bmatrix} 2 \\ -4 \\ -3 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$

11. (a) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

    (b) $\begin{bmatrix} 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 2/\sqrt{3} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{6} \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}$

12. (a) $\{1, t^2 - \frac{2}{3}\}$

    (b) $\{\frac{1}{\sqrt{2}} t\}$

13. $1 + \pi - 2\sin t - \sin 2t$

14. (a) $\begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix}$

    (b) $P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$, $\quad Q' = 3y_1^2 - 3y_2^2$

    (c) indefinite

15. $\begin{bmatrix} 1/3 & 2/\sqrt{5} & -2/\sqrt{45} \\ -2/3 & 1/\sqrt{5} & 4/\sqrt{45} \\ 2/3 & 0 & 5/\sqrt{45} \end{bmatrix} \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

16. $\lambda(\lambda + 1)^2$

17. (a) Span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$, Span $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

    (b) H

    (c) Span $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

18. $P^{-1}AP = J$, where $P = \begin{bmatrix} 0 & 1 & 2 \\ -3 & 1 & 1 \\ 6 & 0 & 6 \end{bmatrix}$, $J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

19. $a \neq 3, 4$

20. $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

21. (a) $\lambda = 0, a + b$

    (b) $E_0 = \text{Span} \left\{ \begin{bmatrix} -b \\ a \end{bmatrix} \right\}$, $\quad E_{a+b} = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

    (c) $a = 1, \ b = -1$

22. $b = -2, \ \frac{9}{2}$

23. $a = c = d = e = 0, \ b = \frac{4}{5}, \ f = \frac{-3}{5}$

24. (a) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

    (b) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

25. If $|z| = 1$ then $|z - w| = |\bar{z}||z - w| = |1 - \bar{z}w|$, and if $|w| = 1$ then $|z - w| = |\bar{w} - \bar{z}| = |\bar{w} - \bar{z}||w| = |1 - \bar{z}w|$.