DDC Final Exam (May 2019)

1. **[2]** Write $z = -5 - 5i$ in polar form.

2. **[2]** Find all complex solutions to $z^4 = -16$.

3. **[3]** Sketch the set $\{z \in \mathbb{C} \colon \operatorname{Re}(z^2) + (\operatorname{Im}(z))^2 = \operatorname{Im}(z)\}$ in the complex plane.

4. **[4]** Let $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2\}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ be bases of a vector space V, and suppose that $\mathbf{a}_1 = \mathbf{b}_2$, $\mathbf{a}_2 = \mathbf{b}_1 + \mathbf{b}_2$. Find

   (a) $[\mathbf{x}]_{\mathcal{B}}$ for $\mathbf{x} = 3\mathbf{a}_1 + 4\mathbf{a}_2$.

   (b) the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{A}$.

5. **[5]** In $\mathbb{P}_1$.

   (a) Find the change-of-coordinates matrix from the basis $\mathcal{B} = \{1, t\}$ to the basis $\mathcal{C} = \{1 + 2t, 3 - t\}$.

   (b) Find the change-of-coordinates matrix from the basis $\mathcal{B} = \{t, 1+4t\}$ to the basis $\mathcal{C} = \{1+2t, 1+t\}$.

6. **[4]** Find the eigenvalues of $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$.

7. **[5]** Define $T : \mathbb{P}_1 \to \mathbb{R}^2$ by $T(p) = \begin{bmatrix} p(2) \\ p(3) \end{bmatrix}$.

   (a) Find the matrix of T relative to the basis $\{1, t\}$ of $\mathbb{P}_1$ and the standard basis of $\mathbb{R}^2$.

   (b) Find the matrix of T relative to the basis $\mathcal{B} = \{1 - 2t, t\}$ of $\mathbb{P}_1$ and the basis $\mathcal{C} = \left\{ \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$ of $\mathbb{R}^2$.

8. **[7]** Let $A = \begin{bmatrix} 0.5 & -0.6 \\ 0.75 & 1.1 \end{bmatrix}$.

   (a) Show that the eigenvalues of A are $\lambda = 0.8 \pm 0.6i$.

   (b) Find the general **real** solution of $\dfrac{d\mathbf{x}}{dt} = A\mathbf{x}$.

   (c) Describe the shapes of typical trajectories.

9. **[3]**

   (a) Find a 2×2 matrix that is diagonalizable but not invertible.

   (b) Find a 2×2 matrix that is invertible but not diagonalizable.

   (c) Find a 2×2 matrix that has eigenvalues 0 and 4 but no zero entries.

10. **[5]** Let $A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$. Use diagonalization to find a formula for A^k where k is a positive integer.

11. **[2]** Show that if $A^2 = 0$ then the only possible eigenvalue of A is 0.

12. **[3]** For this question we're in $\mathbb{R}^2$ with the usual dot product. Is the set orthonormal, orthogonal (but not orthonormal), or neither?

    (a) $\operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

    (b) $\left\{ \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}, \begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix} \right\}$

    (c) $\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

13. **[3]** Assume that A is $m \times 4$ with orthogonal columns of length $1, 3, 2, 5$. What is $A^T A$? Justify your answer.

14. **[6]** Find the orthogonal projection of $\mathbf{y}$ onto $\operatorname{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Use the dot product.

    (a) $\mathbf{y} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

    (b) $\mathbf{y} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

15. **[4]** Find a QR factorization of $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix}$.

16. **[7]** Let $\mathbb{P}_2$ (polynomials of degree at most two) have the inner product $\langle p, q \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$.

    (a) Compute $\langle t + 1, t^2 - t \rangle$.

    (b) Compute $\|t + 1\|$.

    (c) Let $H = \operatorname{Span}\{t + 1\}$. Find an orthogonal basis of $H^\perp$.

17. **[3]** Consider the rectangle of maximum area that fits inside the ellipse $36x_1^2 + x_2^2 = 9$. Use a quadratic form to find an intersection point of this rectangle with the ellipse. Show your work, don't use a memorized formula.

18. **[2]** Let A be a 2×2 real symmetric matrix with two distinct eigenvalues. If $\begin{bmatrix} b \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ 9 \end{bmatrix}$ are eigenvectors of A, what are the possible values of b?

19. **[4]** Find a singular value decomposition of

$$A = \begin{bmatrix} 5 & 5 \\ -1 & 7 \end{bmatrix}$$

    The eigenvalues of $A^T A = \begin{bmatrix} 26 & 18 \\ 18 & 74 \end{bmatrix}$ are 20 and 80.

20. **[6]** Find the minimum polynomial of the given matrix.

    (a) $A = \begin{bmatrix} 4 & 0 \\ 1 & 4 \end{bmatrix}$

    (b) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 \end{bmatrix}$

21. **[3]** Given $H_1 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x - y - z = 0 \right\}$,

$$H_2 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x = 2y \text{ and } y = z \right\}, \quad H_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\},$$

show that the sum $H_1 + H_2 + H_3$ is direct or explain why it is not.

22. **[5]** Find the Jordan canonical form (and corresponding transition matrix) of

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -2 & 1 & 2 \end{bmatrix} \quad \text{Hint: } m_A(\lambda) = \lambda^2(\lambda - 2).$$

23. **[2]** Assume that A is a square matrix with minimum polynomial $m_A(\lambda) = (\lambda - 1)(\lambda + 1)(\lambda - 4)$. Explain why $A^2 - 3A - 4I \neq 0$.

24. **[2]** Given that $z \in \mathbb{C}$ is not real and $|z| = 1$, show that $Re\left(\frac{z-1}{z+1}\right) = 0$.

25. **[2]** Let H_1, H_2, H_3 be subspaces of a vector space V. Is the equation

$$H_1 \cap (H_2 + H_3) = H_1 \cap H_2 + H_1 \cap H_3$$

always true? Give a proof or find a counterexample.

26. **[2]** The least-squares line for the data $(0,0), (1,d), (2,7)$ is $y = -\frac{1}{2} + \frac{7}{2}x$. Find d.

27. **[4]** Answer **exactly two** of the following four questions.

(a) For which value(s) of $d \in \mathbb{R}$ is the 2×2 matrix

$$\begin{bmatrix} d+1 & 1 \\ 3-d & 3 \end{bmatrix}$$

diagonalizable? You may wish to use the Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

(b) Let A be $n \times n$ and diagonalizable with the only eigenvalues being 1 and -1. Show that $A^2 = I_n$.

(c) Orthogonally diagonalize $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ where $b \neq 0$. Clearly state P and D. Hint: start by trying to guess an eigenvector.

(d) Let A be an $n \times n$ orthogonal matrix and let $\lambda \in \mathbb{C}$ be an eigenvalue of A. Show that $|\lambda| = 1$. Hint: calculate $\|A\mathbf{x}\|^2$.

Answers

1. $5\sqrt{2}e^{5\pi i/4}$

2. The parabola $b = a^2$.

3. $2e^{i\phi}$ where $\phi = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$

4. (a) $\begin{bmatrix} 4 \\ 7 \end{bmatrix}$

(b) $\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$

5. (a) $\frac{1}{7}\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix}$

6. $-2, 1$

7. (a) $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

8. (a)

(b) $c_1 \begin{bmatrix} -0.3\cos(0.6t) - 0.6\sin(0.6t) \\ 0.75\cos(0.6t) \end{bmatrix} e^{0.8t}$
$+c_2 \begin{bmatrix} 0.6\cos(0.6t) - 0.3\sin(0.6t) \\ 0.75\sin(0.6t) \end{bmatrix} e^{0.8t}$
(answers will vary)

(c) Sprirals out.

9. (answers will vary)

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 0 \\ 1 - 2^k & 2^k \end{bmatrix}$

11. Hint: start with $A\mathbf{x} = \lambda\mathbf{x}$, multiply both sides by A.

12. (a) neither

(b) orthonormal

(c) orthogonal (but not orthonormal)

13. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 25 \end{bmatrix}$

14. (a) $\begin{bmatrix} 0 \\ 1/2 \\ 1/2 \end{bmatrix}$

(b) $\begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix}$

15. $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{3} \\ 0 & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 3\sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}$

16. (a) 0
    (b) $\sqrt{5}$
    (c) $\{t^2 - t, 5t^2 + t - 4\}$

17. $(1/2\sqrt{2}, 3/\sqrt{2})$

18. $27/4, -4/3$

19. $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 4\sqrt{5} & 0 \\ 0 & 2\sqrt{5} \end{bmatrix} \begin{bmatrix} 1/\sqrt{10} & 3/\sqrt{10} \\ -3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}$

20. (a) $(\lambda - 4)^2$
    (b) $\lambda(\lambda - 1)$
    (c) $(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)(\lambda-6)(\lambda-7)$

21. It is not direct since $H_1 \cap (H_2 + H_3) = H_1 \cap H_2 = H_2 \neq \{\mathbf{0}\}$.

22. $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $P^{-1}AP = J$

23. If it were 0 then $(\lambda - 4)(\lambda + 1)$ would be a degree two polynomial that evaluates to zero when we plug in A. This is impossible since $\deg(m_A) = 3$.

24. $\dfrac{z-1}{z+1} = \dfrac{(z-1)(\bar{z}-1)}{(z+1)(\bar{z}-1)} = ...$ (continue the this calculation; at some point replace $z\bar{z}$ with 1 and then separate into real and imaginary parts; you may wish to replace z with $a + bi$)

25. Not always true. Counterexample: three distinct lines through the origin in $\mathbb{R}^2$.

26. 2

27. (a) $d \neq 4$
    (b) $A^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1} = PIP^{-1} = I$
    (c) $P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$, $D = \begin{bmatrix} a+b & 0 \\ 0 & a-b \end{bmatrix}$
    (d) hint: $\|A\mathbf{x}\|^2 = \|\lambda\mathbf{x}\|^2 = ...$
    and also $\|A\mathbf{x}\|^2 = (A\mathbf{x}) \cdot (A\mathbf{x}) = ...$