DDC Final Exam (May 2022)

1. **[2]** Write $\frac{(1+i)^2}{3-4i}$ *in rectangular form.*

2. **[4]** Find all solutions to $z^4 = z$.
Give any complex answers in exponential form.

3. **[3]** Let $z_1 = e^{\frac{11\pi}{12}i}$ and $z_2 = e^{\frac{2\pi}{3}i}$.
Find $\left(\frac{z_1}{z_2}\right)^{22}$ *in rectangular form.*

4. **[6]** You are given two bases of the plane $x_1 + 2x_2 + 3x_3 = 0$:

$$\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad \mathcal{C} = \left\{ \begin{bmatrix} -7 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ -1 \end{bmatrix} \right\}.$$

Further, let $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Find:

a) the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$;
b) $[\mathbf{x}]_{\mathcal{C}}$ c) $\mathbf{x}$

5. **[4]** Let $A = \begin{bmatrix} 4 & 2 & 0 \\ 1 & 3 & 0 \\ 1 & 1 & 4 \end{bmatrix}$.

a) Find the eigenvalues of A.
b) Find a basis for the eigenspace of the smallest eigenvalue.

6. **[4]** Given $A = \begin{bmatrix} 4 & 5 \\ -1 & 2 \end{bmatrix}$, find P, C such that $A = PCP^{-1}$ where C can be written as the product of a scaling matrix and a rotation matrix. (Provide only P and C... *not* P^{-1}, S or R.)

7. **[3]** Let $\mathbb{R}^3 = H_1 \oplus H_2$, where H_1 is a plane.

a) What is the dimension of H_2?
b) What is the dimension of $H_1 \cap H_2$?
c) True or False: $H_1 \cup H_2$ is a vector space.

8. **[3]** Complete each sentence with **must**, **might** or **cannot**.

a) If W is a subspace of $\mathbb{R}^n$, then $(W^\perp)^\perp$ ____ equal W.
b) The characteristic polynomial of a square matrix A ____ divide the minimum polynomial of A.
c) Let A be a 5×5 matrix. $\text{Col}(A)^\perp$ ____ equal $\text{Row}(A)$.

9. **[2]** Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$. The vector $\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ can uniquely be written $a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3$. Use the fact that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are orthogonal to find a_3.

10. **[3]** You are given the points $(0,0)$, $(0,2)$, $(\ln 2, 5)$. **Find the coefficients** for the equation $y = \beta_0 e^x + \beta_1 e^{-x}$ that best fits the data.

11. **[4]** Use the Gram-Schmidt process to find an **orthonormal** basis for the vector space spanned by $\mathcal{S} = \left\{ \begin{bmatrix} 0 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 16 \\ 8 \\ 2 \end{bmatrix} \right\}$

12. **[4]** Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -2 & 11 \\ 2 & 1 & 2 \end{bmatrix}$.

a) Find the QR factorization of A.
b) If $AB = \begin{bmatrix} 3 & 6 \\ 3 & 6 \\ 3 & 6 \end{bmatrix}$ then find RB without finding B.

13. **[8]** Define $(X, Y) = \text{trace}(X^T Y)$ on $M_{2\times 2}$, and let

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & 0 \\ d & e \end{bmatrix}.$$

a) What is the norm of A?
b) Find (A, B).
c) Find the cosine of the angle between A and B.
d) Find the values of d and e so that C is orthogonal to both A and B.

14. **[6]** Let $A = \begin{bmatrix} -5 & -3 \\ -3 & 3 \end{bmatrix}$.

a) Write the quadratic form $Q(x)$ for A.
b) Write an upper triangular matrix whose quadratic form is the same as that of A.
c) What is the maximum value of Q given the restriction that x is a unit vector?
d) Find a unit vector which achieves the maximum value in part c.
e) A unit vector u is orthogonal to the vector from part d. Find $Q(u)$.

15. **[4]** Find the Jordan canonical form of $A = \begin{bmatrix} 4 & -2 & -1 \\ 0 & 2 & -1 \\ -1 & 3 & 5 \end{bmatrix}$ and the associated transition matrix.

16. **[2]** Let A be a square matrix (not necessarily symmetric). Show that the quadratic form of A equals the quadratic form of A^T.

17. **[2]** Show that a square matrix with **orthonormal** columns must have a determinant equal to 1 or -1.

18. Find a singular value decomposition and the pseudoinverse of $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 2 \end{bmatrix}$.

Answers

1. $-\frac{8}{25} + \frac{6}{25}i$

2. $0, 1, e^{\frac{2\pi}{3}i}, e^{\frac{4\pi}{3}i}$

3. $-i$

4. a) $\begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}$;

b) $\begin{bmatrix} -1 \\ -3 \end{bmatrix}$;

c) $\begin{bmatrix} -8 \\ 1 \\ 2 \end{bmatrix}$.

5. a) $2, 4, 5$; b) $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$.

6. a) 1; b) 0; c) False

7. a) Must; b) Might; c) Cannot

8. $\frac{10}{21}$

9. $\beta_0 = 3, \beta_1 = -2$

10. $\left\{ \frac{1}{3} \begin{bmatrix} 0 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix} \right\}$

11. a) $Q = \frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{bmatrix}$; $R = \begin{bmatrix} 3 & 0 & 9 \\ 0 & 3 & -6 \\ 0 & 0 & 3 \end{bmatrix}$

b) $RB = \begin{bmatrix} 5 & 10 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}$

12. a) $\sqrt{15}$; b) 11; c) $\frac{11}{\sqrt{13}\sqrt{15}}$; d) $d = 5, e = -4$.

13. a) $-5x_1^2 - 6x_1x_2 + 3x^2$

b) $\begin{bmatrix} -5 & -6 \\ 0 & 3 \end{bmatrix}$

c) 4

d) $\frac{1}{\sqrt{10}} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

e) -6

14. $J = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$; $P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ -2 & -1 & -1 \end{bmatrix}$

15. Since $x^T Ax$ is a 1×1 matrix,

$$x^T Ax = (x^T Ax)^T = x^T A^T x,$$

as required.

16. If A is square and $A^T A = I$ then

$$1 = \det(I) = \det(A^T A) = \det(A^T)\det(A) = (\det(A))^2,$$

which implies that $\det(A) = \pm 1$, as required.

17. $A = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 & -\sqrt{5} & 0 \\ 1 & 0 & -2 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$;

$A^\dagger = \frac{1}{10} \begin{bmatrix} 5 & 1 & 2 \\ -5 & 1 & 2 \end{bmatrix}$