201-NYB-05: Calculus II Science
Final Examination - Winter 2022

Problems:

1. (30 points) Evaluate the following integrals.

(a) $\displaystyle\int_0^{\pi/4} \tan^4\theta \sec^4\theta \, d\theta$ (b) $\displaystyle\int (\ln x)^2 \, dx$

(c) $\displaystyle\int \frac{\sec x \tan x}{\sqrt{9-\sec^2 x}} \, dx$ (d) $\displaystyle\int_1^2 \sqrt{3+2x-x^2} \, dx$

(e) $\displaystyle\int \frac{x^4+3x^2+8}{x^4+4x^2} \, dx$

2. (6 points) Evaluate the following limits. If using l'Hospital's rule, justify why it may be used.

(a) $\displaystyle\lim_{x\to 1} \frac{\sin(\ln x) - x + 1}{(x-1)^2}$ (b) $\displaystyle\lim_{x\to-\infty} (1+e^x)^{e^{-x}}$

3. (9 points) For each of the following improper integrals, either evaluate it or show that it diverges.

(a) $\displaystyle\int_0^{\pi/2} \cot x \, dx$ (b) $\displaystyle\int_0^{\infty} \frac{x}{e^x} \, dx$

4. (a) (4 points) Sketch the region $\mathcal{R}$ bounded by the curves $y=\sqrt{x}$, $x = 2-y^2$ and the x-axis, and find its area.

(b) (3 points) Set up, but **do not evaluate** an integral representing the volume of the solid obtained by rotating the region $\mathcal{R}$ from part (a) about the line $x=2$.

5. (6 points) Solve the initial value problem

$$\frac{y'}{y} - 2x\sqrt{y^2-1} = 0, \quad y(0)=2.$$

Express y explicitly as a function of x.

6. According to Newton's law of cooling, the rate of change dT/dt of the temperature of an object is proportional to the difference between its temperature T and the ambient temperature T_A. Suppose that the ambient temperature of a large room is 20°C, that the initial temperature of a small object in that room is 100°C, and that after 1 minute the object is observed to have cooled to 60°C.

(a) (4 points) Set up a differential equation representing this situation and solve the initial value problem.

(b) (1 point) How long will it take for the object to cool to 30°C? For full marks, give the simplified exact answer.

7. (4 points) Find the length of the curve $y=\ln(\cos x)$ between $x=0$ and $x=\pi/4$.

8. (6 points) For each of the following series, either find its sum, or show that it diverges. Justify your answers.

(a) $4 - 3 + \frac{9}{4} - \frac{27}{16} + \frac{81}{64} - \cdots$ (b) $\displaystyle\sum_{n=2}^{\infty} \left(e^{1/(n+1)} - e^{1/(n-1)}\right)$

9. (6 points) Determine whether each of the following series converges or diverges. Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan(n)}{1-e^{-n}}$ (b) $\displaystyle\sum_{n=2}^{\infty} \frac{3-\cos^2 n}{n-1}$

10. (9 points) Determine whether each of the following series converges absolutely, converges conditionally, or diverges. Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{3^n\cdot(2n)!}$ (b) $\displaystyle\sum_{n=1}^{\infty} (-2n)^n[\sin(1/n)]^n$

(c) $\displaystyle\sum_{n=3}^{\infty} (-1)^n \frac{n}{n^2+4}$

11. (2 points) If a series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges **conditionally** and $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ exists, what can we say about $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$? Justify your answer.

12. (5 points) Find the radius and interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^3\cdot 2^{n+1}}$.

13. (5 points) Find the Taylor series for $f(x) = (1-2x)^{-1}$ centred at $x=-1$.

Answers:

1. (a) 12/35 (b) $x(\ln x)^2 - 2(x\ln x - x) + C$

(c) $\arcsin\left(\frac{\sec x}{3}\right) + C$ (d) $\frac{2\pi + 3\sqrt{3}}{6}$

(e) $x - \frac{2}{x} - \frac{3}{2}\arctan\left(\frac{x}{2}\right) + C$

2. (a) $-1/2$ (b) e

3. (a) Diverges (to ∞). (b) Converges to 1.

4. (a) 4/3

(b) $V = \pi \displaystyle\int_0^1 \left((2-y^2)^2 - y^4\right) \, dy$

5. $y = \sec(x^2 + \pi/3)$

6. $\ln(\sqrt{2}+1)$

7. (a) $T = 80\cdot\left(\frac{1}{2}\right)^t + 20$

(b) 3 minutes.

8. (a) 16/7 (GST) (b) $2 - e - \sqrt{e}$ (TS)

9. (a) D (TFD/nTT) (b) D (DCT)

10. (a) AC (RT) (b) D (RoT)

(c) CC (LCT, AST)

11. By the ratio test, if $\lim|a_{n+1}/a_n| < 1$ the series would converges **absolutely**, whilst if $\lim|a_{n+1}/a_n| > 1$, the series would diverge. Since neither is the case, the only remaining possibility is that $\lim|a_{n+1}/a_n| = 1$.

12. $R=2, \quad I=[0,4]$

13. $T(x) = \displaystyle\sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}}(x+1)^n$