Exponent Rules and Simplification

Understanding the Exponent Rules

$a^m a^n = a^{(m+n)}$	$\frac{a^m}{a^n} = a^{(m-n)} = \frac{1}{a^{(n-m)}}$	$(a^m)^n = (a^n)^m = a^{mn}$
$\sqrt[n]{a^m} = (\sqrt[n]{a})^m = a^{m/n}$ $\sqrt{a^m} = (\sqrt{a})^m = a^{m/2}$	$a^{-m} = \frac{1}{a^m}$ OR $\frac{1}{a^{-n}} = a^n$ $\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m$	$a^m b^m = (ab)^m$

1. $a^m a^n = a^{(m+n)}$ | $x^3x^4 = x^{3+4} = x^7$

To find the product of two numbers with the same base, just add the exponents.
I find it useful to think about the number of things you have in total. For example, consider x^3x^4. The x^3 means I have 3 xs, and the x^4 means I have 4 more xs. So 7 in total.
In other words, $x^3x^4 = (x \cdot x \cdot x)x^4 = (x \cdot x \cdot x)(x \cdot x \cdot x \cdot x) = x^7$.

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2. $\frac{a^m}{a^n} = a^{(m-n)} = \frac{1}{a^{(n-m)}}$ | $\frac{6x^5}{x^3} = 6x^{5-3} = 6x^2$, $\frac{6x^5}{x^3} = \frac{6}{x^{3-5}} = \frac{6}{x^{-2}}$

To find the quotient of two numbers with the same base, subtract the exponent of the denominator from the exponent of the numerator.
Again, it is useful to think about the number of things you have in total. For example, consider $\frac{6x^5}{x^3} = 6x^{5-3} = 6x^2$. The x^5 in the numerator means I have 5 xs in the numerator. The x^3 in the denominator means that I have 3 xs in the denominator to take away from the 5 in the numerator. So there will be 2 left over. In other words, $\frac{6x^5}{x^3} = \frac{6 \cdot \not{x} \cdot \not{x} \cdot \not{x} \cdot x \cdot x}{\not{x} \cdot \not{x} \cdot \not{x}} = 1 \cdot 1 \cdot 1 \cdot 6 \cdot x \cdot x = 6x^2$
In this case, it is important to remember that $\frac{x}{x} = 1$.

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3. $(a^m)^n = (a^n)^m = a^{mn}$ | $(x^4)^2 = (x^2)^4 = x^{4 \cdot 2} = x^8$

If we put a power to a power, we must multiply the two numbers together, and it does not matter which one comes first.
In this case, we can think of this as x^4 being squared, so $(x^4)^2 = (x^4)(x^4) = (x \cdot x \cdot x \cdot x)(x \cdot x \cdot x \cdot x) = x^8$, and we can see that there are 8 xs in total here. Note that this is the same thing as saying x^2 to the power of 4, so $(x^2)^4 = (x^2)(x^2)(x^2)(x^2) = (x \cdot x)(x \cdot x)(x \cdot x)(x \cdot x) = x^8$. Once again, this results in 8 xs.

4.	$\sqrt[n]{a^m} = (\sqrt[n]{a})^m = a^{m/n}$ $\sqrt{a^m} = (\sqrt{a})^m = a^{m/2}$	$\sqrt[4]{x^2} = (\sqrt[4]{x})^2 = x^{2/4} = x^{1/2}$

If a root is raised to a fractional exponent, the numerator is the power and the denominator is the root. It doesn't matter if you do the power or the root first. Thinking of this with numbers instead of variables, consider $4^{3/2}$. This is saying that we want to take the square root of 4 and put it to the power of 3. This is the same thing as putting 4 to the power of 3 and then taking the square root. So $4^{3/2} = (\sqrt{4})^3 = 2^3 = 8$ OR $4^{3/2} = \sqrt{4^3} = \sqrt{64} = 8$.
It may be useful to remember that $\sqrt{x} = y$ means that $y^2 = a$. In other words, it answers the question: what can be multiplied by itself two times to result in x. In the same way, $\sqrt[n]{x} = y$ means that $y^n = a$. In other words, it answers the question: what can be multiplied by itself n times to result in x.

5.	$a^{-m} = \frac{1}{a^m}$ OR $\frac{1}{a^{-n}} = a^n$ $\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m$	$4^{-2} = 14^2 = \frac{1}{16}$ $\left(\frac{2}{5}\right)^{-3} = \left(\frac{5}{2}\right)^3 = \frac{5^3}{2^3} = \frac{125}{8}$

It is important to remember that a negative exponent is just notation meaning take the reciprocal.
It is also important to remember that we ONLY take the reciprocal of the part with the negative exponent. So, for example $5x^{-3} = \dfrac{5}{x^3}$ because only the x is to the power of -3. Or $\dfrac{2}{3x^{-6}} = \dfrac{2}{3}x^6$. The 3 stays in the denominator because it is NOT to the power of -6. However, $(5x)^{-3} = \dfrac{1}{(5x)^3} = \dfrac{1}{125x^3}$ because in this case both the x AND the 5 are to the power of -3. Or $\dfrac{2}{(3x)^{-2}} = 2(3x)^2 = 2 \cdot 3^2 \cdot x^2 = 18x^2$ because in this case BOTH the x and the 3 are to the power of -2.

6.	$a^m b^m = (ab)^m$	$(2^3)(3^3) = 6^3 = 216$

In this case, the BASES are different, but the exponents are the same. When this occurs, we can multiply the bases together BEFORE applying the exponent. To understand this, remember that multiplication is commutative (order doesn't matter), and consider $(2^3)(3^3) = (2 \cdot 2 \cdot 2)(3 \cdot 3 \cdot 3) = (2 \cdot 3)(2 \cdot 3)(2 \cdot 3) = (6)(6)(6) = 6^3 = 216$

Simplifying Expressions Using the Exponent Rules

To simplify expressions using the exponent rules, remember:

1. **The only possible cancellation between the numerator and denominator of a fraction is that of a common factor**

DO NOT DO THIS! $\frac{2x+3}{2x+4} = \frac{\cancel{2x}+3}{\cancel{2x}+4} = \frac{3}{4}$ I'm serious, every time you do this, a puppy dies. In this case, you canceled a common **term** (separated by addition, not multiplication) and not a common **factor**. Totally not allowed.

THIS IS OKAY: $\frac{2x(x+3)}{(x^2+1)(2x)} = \frac{\cancel{2x}(x+3)}{(x^2+1)\cancel{2x}}$ Yay! You canceled a common **factor** (because everything was separated by multiplication). The puppies are safe.

2. **Denominators can be distributed to the different terms in a numerator, but numerators CANNOT be distributed to the different terms in the denominator.**

DO NOT DO THIS! $\frac{\sqrt{x}}{x^2+2x+1} = \frac{\sqrt{x}}{x^2} + \frac{\sqrt{x}}{2x} + \frac{\sqrt{x}}{1} = x^{1/2-2} + \frac{1}{2}x^{1/2-1} + \sqrt{x} = x^{-3/2} + \frac{1}{2}x^{-1/2} + \sqrt{x}$
Seriously, don't do it. Here, you distributed the numerator to the different terms of the denominator. I mean, I guess at least after you used your exponent rules correctly, but that first step was totally not allowed.

THIS IS OKAY: $\frac{x^2+2x+1}{\sqrt{x}} = \frac{x^2}{\sqrt{x}} + \frac{2x}{\sqrt{x}} + \frac{1}{\sqrt{x}} = x^{2-1/2} + 2x^{1-1/2} + x^{-1/2} = x^{3/2} + x^{1/2} + x^{-1/2}$
The denominator was distributed to the different terms in the numerator, so you're all good there. And then the exponent rules were all used perfectly. Great job!

3. **Be careful: When distributing the denominator to each of the terms in the numerator, every TERM gets the denominator only ONE time.**

So, for example, in $\frac{x^2+3x+xe^x}{\sqrt{x}}$, the numerator has only 3 terms: $x^2, 3x$, and xe^x.

GOOD: $\frac{x^2+3x+\sqrt{x}e^x}{\sqrt{x}} = \frac{x^2}{\sqrt{x}} + \frac{3x}{\sqrt{x}} + \frac{\sqrt{x}e^x}{\sqrt{x}} = x^{3/2} + 3x^{1/2} + e^x$

BAD: $\frac{x^2+3x+xe^x}{\sqrt{x}} = \frac{x^2}{\sqrt{x}} + \frac{3x}{\sqrt{x}} + \frac{\sqrt{x}}{\sqrt{x}}\frac{e^x}{\sqrt{x}}$

4. **Exponents CANNOT be distributed in addition and subtraction.** $(a+b)^n \neq a^n + b^n$ **Also remember:** $(a+b)^2 = a^2 + 2ab + b^2$)

DO NOT DO THIS: $(x+3)^3 = x^3 + 3^3$ Noooooooo! Please don't do this! Are you trying to make me cry? $(x+3)^3$ means $(x+3)(x+3)(x+3) = (x+3)(x^2+6x+9) = x^3+6x^2+9x+3x^2+18x+27 = x^3+9x^2+27x+27$

ALSO NOT OKAY: $\sqrt{25+9} = \sqrt{25} + \sqrt{9} = 5+3 = 8$. Just no. I mean, think about it. $25+9 = 34$, so $\sqrt{25+9} = \sqrt{34}$ which definitely does NOT equal 8. Roots are powers, too (see the exponent rules), so they definitely can't be distributed.

THIS IS OKAY: $(2x+3)^2 = (2x+3)(2x+3) = (2x)^2 + (2x)(3) + (3)(2x) + 3^2 = 4x^2 + 12x + 9$ Yes, we squared the first and last terms, but we ALSO multiplied the inner terms and the outer terms. We applied FOIL: First, Outer, Inner, Last. So this is totally okay.

Simplifying Exercises

Most of the time in Calculus, the goal will be to write things as kx^n, if possible. So we want to simplify any numbers and bring them to the front, then combine all the xs (and other variables, if there are any).
Some examples to combine all the things we've just learned (try to think of which of the above rules are being used). You may **find the solution on the next page**.

Exercise 1 Simplify $(3x^3y^2)^2$

Exercise 2 Simplify $x^2x^7y^3y^{-1}$

Exercise 3 Simplify $\frac{x^{-4}}{x^{-13}}$

Exercise 4 Simplify $\sqrt{50x^4}$

Exercise 5 Simplify $6x^2 + \frac{4x^4 - \sqrt{x}}{x^2}$

Exercise 6 Simplify $\frac{6\sqrt{x} - 5x^2}{30\sqrt[3]{x}}$

Simplifying Solutions

Be sure to try the exercises yourself before immediately looking at the solutions. If you're still confused, be sure to watch the theory and solutions videos.

Solution 1 $(3x^3y^2)^2 = (3^2)(x^3)^2(y^2)^2 = 9x^6y^4$
First, we apply Rule 6 to distribute the power of 2 inside the multiplication. Then, we apply Rule 3 to find the new powers of x and y.

Solution 2 $x^2x^7y^3y^{-1} = x^{2+7}y^{3-1} = x^9y^2$
We apply Rule 1 to combine the xs and ys.

Solution 3 $\dfrac{x^{-4}}{x^{-13}} = \dfrac{x^{13}}{x^4} = x^{13-4} = x^9$
We first remove the negative exponents by flipping the fraction, using Rule 5. Then we apply Rule 2 to combine the xs.
We could also think of this as $\dfrac{x^{-4}}{x^{-13}} = x^{-4-(-13)} = x^9$ and just immediately apply Rule 2.

Solution 4 $\sqrt{50x^4} = \sqrt{5 \cdot 5 \cdot 2 \cdot x^2 \cdot x^2} = 5\sqrt{2}x^2$
In order to find $\sqrt{50}$, we factor it into its prime factors, so $50 = 5 \cdot 5 \cdot 5 \cdot 2$. The 5 occurs 2 times, so we can see that $\sqrt{50} = 5\sqrt{2}$. Then we see that $\sqrt{x^4} = \sqrt{x^2 \cdot x^2} = x^2$.
We could also apply Rule 4 and think of this as $\sqrt{x^4} = x^{4/2} = x^2$.

Solution 5 $6x^2 + \dfrac{4x^4 - \sqrt{x}}{x^2} = 6x^2 + \dfrac{4x^4}{x^2} - \dfrac{\sqrt{x}}{x^2} = 6x^2 + 4x^{4-2} - x^{1/2-2} = 6x^2 + 4x^2 - x^{-3/2} = 10x^2 + x^{-3/2} = 10x^2 + \dfrac{1}{x^{3/2}}$
In the second term, there are 2 terms in the numerator and 1 in the denominator, so the denominator can be distributed to the numerator. We then combine the xs using Rule 2.
We see that there are two terms in which the variables is x^2, so we combine like terms to get $6x^2 + 4x^2 = 10x^2$.
Depending on the purpose of the simplification, leaving the last term as $x^{-3/2}$ may be more useful than using Rule 5 to get $\dfrac{1}{x^{3/2}}$.

Solution 6 $\dfrac{6\sqrt{x} - 5x^2}{30\sqrt[3]{x}} = \dfrac{6x^{1/2}}{30x^{1/3}} - \dfrac{5x^2}{30x^{1/3}} = \dfrac{6}{30}x^{1/2-1/3} - \dfrac{5}{30}x^{2-1/3} = \dfrac{1}{5}x^{1/6} - \dfrac{1}{6}x^{5/3}.$
Since there are 2 terms in the numerator and only 1 in the denominator, we can distribute the denominator to the numerator. At the same time, we applied Rule 4 to rewrite the roots as x to a power. We then applied Rule 2.