Factoring

Factoring Monic Quadratics using the Product-Sum Technique

A **monic polynomial** is a polynomial whose leading coefficient is 1. That is, the variable with the highest power will ONLY be multiplied by 1.
A **quadratic polynomial** is a polynomial whose highest power is 2.

Technique: $x^2+bx+c = (x+m)(x+n)$ where $m \cdot n = c$ and $m+n = b$

Exercise Factor the monic quadratic polynomial $x^2 - 3x - 28$

Solution $x^2 - 3x - 28 = (x-7)(x+4)$ because $\mathbf{-7} \cdot \mathbf{4} = -28$ and $\mathbf{-7} + \mathbf{4} = -3$

Some Tips:

1. If $b^2 - 4ac < 0$, the polynomial is irreducible/cannot be factored. In this case, we consider $a = 1$. If you are unsure, it is always useful to check.
2. If the constant term is positive, then the two values, m and n will either both be positive or both be negative. (As their product needs to result in a positive.
3. If the constant term is negative, as in this example, then one of the terms will be positive and one of the terms will be negative. (As their product needs to result in a negative.) If the coefficient of x is positive, the resulting sum will need to be positive, so the larger value will be positive. If the coefficient of x is negative, as in this example, then the resulting sum will need to be negative, so the larger value will be negative.
4. If you have trouble finding the two values, start by dividing the constant term by 1.
   $(1)(-28) = -28$ but $1 - 28 = -27$ $(2)(-14) = -28$ but $2 - 14 = -12$
   28 is not divisible by 3
   $(4)(-7) = -28$ and $4 - 7 = -3$, so these are our two numbers.

Note: We could also factor $x^4 - 20x^2 + 100$ using the rule $x^2 + bx + c = (x+m)(x+n)$.
Think of it as $(x^2)^2 - 20(x^2) + 100$. We need two numbers whose product is 100 and whose sum is -20, so we can see that the two numbers must be negative. These two numbers will be -10 and -10. We then use x^2 in the formula instead of x, so we get
$x^4 - 20x^2 + 100 = (x^2 - 10)(x^2 - 10) = (x^2 - 10)^2$. Technically, this can be factored even more using the difference of squares, to give
$x^4 - 20x^2 + 100 = (x^2 - 10)(x^2 - 10) = (x^2 - 10)^2 = (x - \sqrt{10})^2(x + \sqrt{10})^2$

Sum and Difference of Squares

Difference of Squares Formula $a^2 - b^2 = (a-b)(a+b)$

Exercise Factor $16x^2 - 81y^2 = (4x)^2 = (9y)^2$

Solution $16x^2 - 81y^2 = (4x)^2 = (9y)^2 = (4x-9y)(4x+9y)$

In this case, we directly applied the difference of squares formula, with $a = \sqrt{16x^2 = 4x}$ and $b = \sqrt{81y^2} = 9y$

Tip: If there is a constant term, it does not necessarily have to be a perfect square. See the following example.

Exercise Factor $4x^2 - 10$

Solution $4x^2 - 10 = (2x)^2 - (\sqrt{10})^2 = (2x - \sqrt{10})(2x + \sqrt{10})$

Once again, we directly applied the difference of squares formula, with $a = \sqrt{4x^2 = 2x}$ and $b = \sqrt{10}$

Sum of Squares A sum of squares, $a^2 + b^2$ is prime and thus cannot be factored/is irreducible

Exercise: Factor $x^2 + 9$

Solution $x^2 + 9$ is a sum of squares. It is therefore irreducible and cannot be factored.

Sum and Difference of Cubes

Sum of Cubes Formula $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

Exercise Factor $8x^3 + 27y^3$

Solution $8x^3 + 27y^3 = (2x)^3 + (3y)^3 = (2x + 3y)(4x^2 - 6xy + 9y^2)$

We directly applied the sum of cubes formula with $a = \sqrt[3]{8x^3} = 2x$ and $b = \sqrt[3]{27y^3} = 3y$

Difference of Cubes Formula $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Exercise Factor $64x^3 - 1$

Solution $64x^3 - 1 = (4x)^3 - (1)^3 = (4x - 1)(16x^2 + 4x + 1)$

We directly applied the difference of cubes formula with $a = \sqrt[3]{64x^3} = 4x$ and $b = \sqrt[3]{1} = 1$

Greatest Common Factor

The **Greatest Common Factor**, or **GCF**, is the largest factor that is shared by all terms. Factoring out the GCF should always be your first step when factoring any polynomial.

Steps for Factoring out the GCF

1. Think of each term in prime factored form (if necessary/useful, write each term in prime factored form)
2. Identify the factors common in all terms.
3. Factor it out by writing it in front and dividing each term by the GCF.
4. After factoring out the GCF, always check to see if more factoring can be done.

Exercise Factor $90x^5y^4 - 45x^3y^5 + 30x^4y^3 + 15x^3y^3$ by first identifying the GCF.

Solution $90x^5y^4 - 45x^3y^5 + 30x^4y^3 + 15x^3y^3 =$
$(5)(3)(3)(2)x^5y^4 - (5)(3)(3)x^3y^5 + (5)(3)(2)x^4y^3 + (5)(3)x^3y^3 = (5)(3)x^3y^3[(3)(2)x^2y - 3y^2 + 2x + 1] =$
$15x^3y^3(6x^2y - 3y^2 + 2x + 1)$

The GCF is $15x^3y^3$. The resulting polynomial after factoring cannot be further factored.

Tips:

1. For variables, if the same variable occurs in every term, we always take the one with the SMALLEST power for the GCF, even if that power is negative or fractional.
2. When dividing each term by the GCF, I find it useful to ask 'What did I take away? So what's left over?'
3. Remember, if one term is the entirety of the GCF, then when factoring out the GCF, there will be a 1 left over.

Factoring by Grouping

Factoring by grouping is useful for cubic polynomials (power of 3) with an even number of terms or for factoring non-monic quadratic polynomials.

Steps for Factoring by Grouping

1. Separate into groups of two terms
2. Factor out the greatest common factor from each of the groups
3. Factor out common factor of the whole, where each group is considered a term

Exercise Factor $2x^3 + 4x^2 + 4x + 8$

Solution $2x^3 + 4x^2 + 4x + 8 = (2x^3 + 4x^2) + (4x + 8) = 2x^2(x + 2) + 4(x + 2) = (x + 2)(2x^2 + 4) = (x + 2)(2)(x^2 + 2) = 2(x + 2)(x^2 + 2)$

OR GCF first

$2x^3 + 4x^2 + 4x + 8 = 2(x^3 + 2x^2 + 2x + 4) = 2((x^3 + 2x^2) + (2x + 4)) = 2(x^2(x + 2) + 2(x + 2) = 2(x + 2)(x^2 + 4))$

Factoring Non-Monic Quadratic Polynomials

Remember, a **monic polynomial** is a polynomial whose leading coefficient is 1. So a non-monic quadratic polynomial will be one in which the highest power of x is 2, and the x^2 term is multiplied by a number other than 1.

Technique $ax^2 + bx + c = ax^2 + mx + nx + c$ where $m \cdot n = a \cdot c$ and $m + n = b$. This can now be factored by grouping.

Exercise Factor $12x^2 + 16x + 5$ by grouping.

Solution We want two numbers whose product is $(12)(5) = 60$ and whose sum is 16. These two numbers are 6 and 10, so $12x^2 + 16x + 5 = (12x^2 + 6x) + (10x + 5) = 6x(2x + 1) + 5(2x + 1) = (2x + 1)(6x + 5)$

Tips:

1. If $b^2 - 4ac < 0$, the polynomial is irreducible/cannot be factored. If you are unsure, it is always useful to check.
2. The order in which you put m and n does not matter. So $ax^2 + mx + nx + b$ will factor in the same way as $ax^2 + nx + mx + b$.
3. If there is a GCF, always start by factoring it out. The resulting polynomial will be easier to factor.
4. If you have trouble finding the two values, start by dividing the constant term by 1.
   $(1)(6) = 60$ but $1 + 60 = 61$ $(2)(30) = 60$ but $2 + 30 = 32$
   $(3)(20) = 60$ but $3 + 20 = 23$
   $(4)(15) = 60$ but $4 + 15 = 19$
   $(5)(12) = 60$ but $5 + 12 = 17$
   $(6)(10) = 60$ and $6 + 10 = 16$, so these are our two numbers.