Logarithms

Important! Logarithms have an **inverse relationship** with exponentials. So make sure you understand exponential functions before tackling logarithms.

Intro to Logarithms

We know the basic version of the exponential function: $y = a^x$, and that the number chosen for a is called the **base** of the exponent.

The **logarithm** with base a (which we write as $\log_a()$) is the **inverse** of the exponential function with base a.

Inverse Relationship between Exponents and Logarithms

$$y = a^{x} \xleftrightarrow[\text{is equivalent to}]{} \log_a y = x$$

For a video explanation, click here.

For example, $2^{3} = 8$ is equivalent to $\log_2 8 = 3$. One way to think about this is to say to yourself that the logarithm is asking the question "what exponent do I need to raise 2 to in order to get 8?"

How to Interpret a Logarithm

$\log_a b =?$ is asking $a^{\boxed{?}} = b$

For a video explanation, click here.

With that in mind, try this quick exercise:

Exercise 1

(a) Rewrite $\log_2(64)$ as an exponential equation, then find its value.

(b) Rewrite $\log_4(8)$ as an exponential equation, then find its value.

(c) Rewrite the exponential property $a^0 = 1$ (for any $a \neq 0$) as a logarithmic equation.

Solutions

(a) $\log_2(64)$ is asking $2^{\boxed{?}} = 64$. $2^6 = 64$, so $\log_2(64) = \mathbf{6}$.

(b) $\log_4(8)$ is asking $4^{\boxed{?}} = 8$. I know that $2^3 = 8$, and that $\sqrt{4} = 4^{1/2} = 2$. So then $\left(\sqrt{4}\right)^3 = \left(4^{1/2}\right)^3 = 4^{3/2} = 8$. $4^{3/2} = 8$, so the answer is $\frac{3}{2}$.

(c) $a^0 = 1$ is equivalent to $\log_a 1 = 0$. **So the logarithm of 1 is always equal to 0.**

Because logarithms and exponents are inverse operations, they have the effect of “undoing" each other (*if* the bases are the same.)

How Logarithms and Exponents 'cancel out'

$$\log_a(a^{z}) = z$$
applying the logarithm of the same base 'undoes' the exponential function

$$a^{(\log_a z)} = z$$
applying the exponential function with the same base 'undoes' the log function

Special case, when the base is e:

$$\ln(e^{z}) = z \qquad e^{(\ln(z))} = z$$

This method of applying the inverse function is essential when we want to solve equations involving trickier exponents or logarithms. Here's an example:

Example
Suppose we want to solve for x in $2^x = 5$. We want to isolate x, but right now it's locked up in the exponential function. To get it out, we 'undo' the exponential function by applying its inverse, which is $\log_2()$:

$$\begin{aligned} 2^x &= 5 && \text{apply } \log_2 \text{ on both sides of the equation} \\ \log_2(2^x) &= \log_2(5) && \text{we know } \log_2(2^x) = x \\ x &= \log_2(5) \end{aligned}$$

Try using these properties and the properties of exponents to solve the next exercise:

Exercise 2

(a) Solve for x in the equation: $5(3^x) = 6$.

(b) Simplify the expression $\ln(e^4 \cdot e^3)$

(c) Simplify the expression $\log_2\left(\frac{1}{\sqrt[3]{2}}\right)$

Solutions

(a) You need to isolate the exponential expression before you can use the property we just learned, so divide both sides by 5, then apply $\log_3()$ to each side:

$$\begin{aligned} 3^x &= \frac{6}{5} \\ \log_3(3^x) &= \log_3\left(\frac{6}{5}\right) \\ x &= \log_3\left(\frac{6}{5}\right) \end{aligned}$$

(b) $\ln(e^4 \cdot e^3) = \ln(e^{4+3}) = \ln(e^7) = \mathbf{7}$.

(c) $\frac{1}{\sqrt[3]{2}} = \left(\sqrt[3]{2}\right)^{-1} = \left(2^{1/3}\right)^{-1} = 2^{-\frac{1}{3}}$, so then $\log_2\left(\frac{1}{\sqrt[3]{2}}\right) = \log_2\left(2^{-\frac{1}{3}}\right) = -\frac{1}{3}$

Properties of Logarithms

Since the logarithm is the **inverse** of the exponential function, the roles of the *input* and the *output* are swapped. You can see this when you compare the properties of logarithms to the properties of exponents – the operations inside and outside the functions are swapped.

Logarithmic Properties	**Corresponding Exponential Properties**
1. $\log_a(x \cdot y) = \log_a(x) + \log_a(y)$	1. $a^{x+y} = a^x \cdot a^y$
2. $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$	2. $a^{x-y} = \frac{a^x}{a^y}$
3. $\log_a(x^y) = y \log_a(x)$	3. $a^{xy} = (a^x)^y$
4. $\log_a(1) = 0$ for any a other than 0	4. $a^0 = 1$ for any a other than 0
5. $\log_a(a) = 1$ for any a	5. $a^1 = a$ for any a

These properties will be essential to know in your Calculus class. They also give us some new ways to solve exponential equations.

Example We could use properties of logarithms to solve each equation from Exercise 2 in another way. In (a), would could start by applying the natural logarithm ($\ln$, which is $\log_e$) to both sides of the equation:

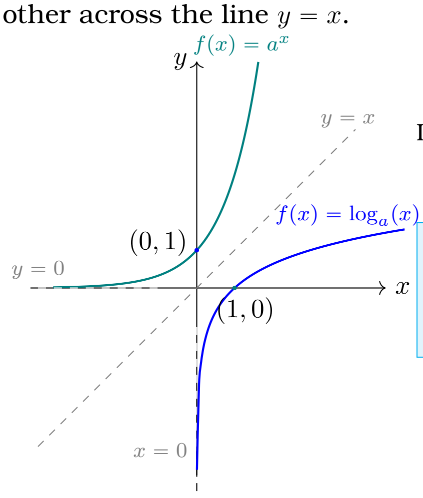
$$\ln(5(3^x)) = \ln(6)$$
use property 1 on the left-hand side
$$\ln(5) + \ln(3^x) = \ln(6)$$
use property 3 to bring x out of the logarithm
$$\ln(5) + x\ln(3) = \ln(6)$$
solve for x
$$x = \frac{\ln(6)-\ln(5)}{\ln(3)}$$

This answer is equivalent to our previous answer of $\log_3\left(\frac{6}{5}\right)$.

Your turn: Try to find a second way to solve (b) and (c), using the properties of logarithms.

Graphs

Here are the graphs of $y = a^x$ and $y = \log_a(x)$ plotted together. Once again, notice that the roles of x (the input) and of y (the output) are swapped. As a result, graphs are mirror images of each other across the line $y = x$.



In Calculus class, it's important in particular to know these two graphs. Comparing the graphs of these two inverse functions we can notice a few more important properties:

6. $\log_a(x)$ only accepts positive x-values	6. a^x only gives positive x-values
7. $\log_a(x)$ has a vertical asymptote at $x = 0$	7. a^x has a horizontal asymptote at $y = 0$