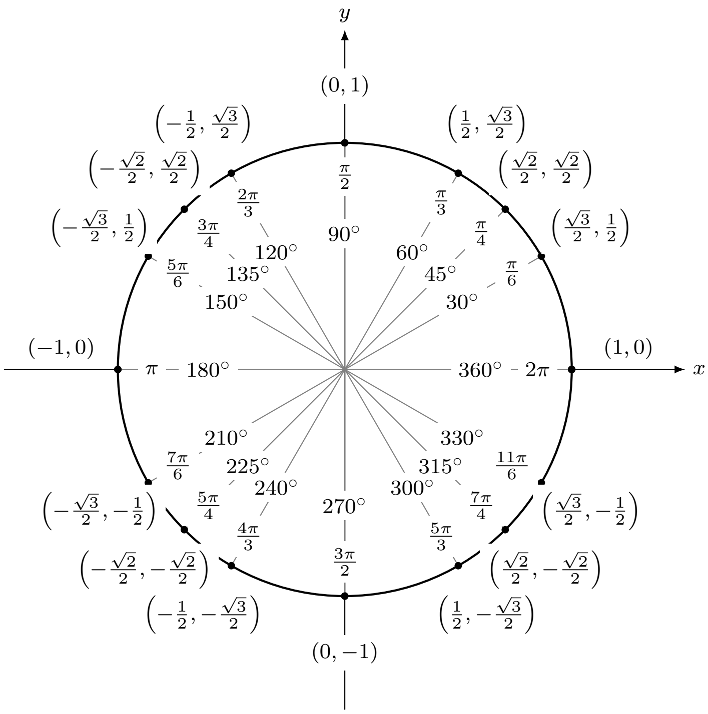
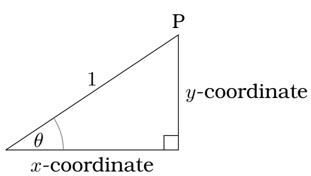
Trigonometry - The Unit Circle

For a video explanation, click here.

Here is the version of the unit circle that you've probably seen before:



The unit circle is a circle of radius 1, centered at $(0,0)$. It is used as a way to **keep track of sine and cosine values of important angles**. If you take any point P around the the unit circle and form the right triangle connecting that point to the origin and to the x−axis, you have:

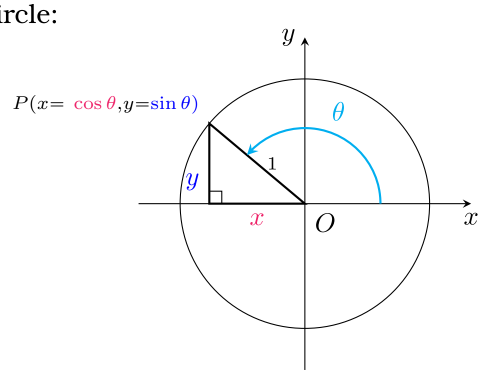


$$\sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{y\text{-coordinate}}{1} = y\text{-coordinate}$$

and

$$\cos(\theta) = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{x\text{-coordinate}}{1} = x\text{-coordinate}$$

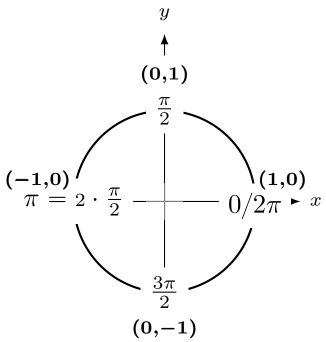
So for any point P on the unit circle, the y-coordinate is equal to $\sin(\theta)$ and the x-coordinate is equal to $\cos(\theta)$, where θ is the angle between the positive x−axis and the line which corrects the point P to the center of the circle:



In the *Science* program at John Abbott, you will need to be able to find the values of the trigonometric functions at these important angles *without* using a calculator or notes. To find them using the unit circle, we separate the job into two tasks: first locate the angle on the circle, and then determine the x (cos) or y (sin)-value.

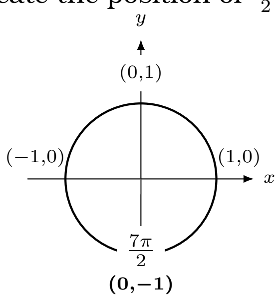
When you count angles around the unit circle, you always **start on the positive** x**−axis**, and go **counterclockwise** (unless the angle is negative, in which case you go the opposite way.)

One full turn around the circle is 2π radians, which means that each half-circle has π radians. That means that each *quarter* circle has $\frac{\pi}{2}$ radians:



We can know the coordinates of these points from the fact that the circle has radius 1. For any multiple of $\frac{\pi}{2}$, simply count around the circle counter-clockwise for the corresponding number of quarter-circles.

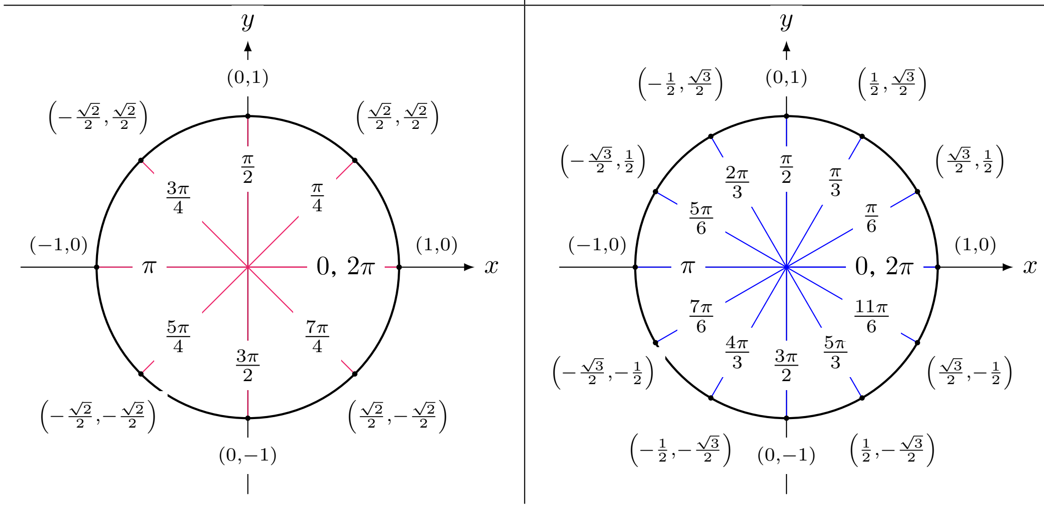
Exercise Locate the position of $\frac{7\pi}{2}$ on the unit circle, and find $\sin\left(\frac{7\pi}{2}\right)$.
Solution



$$\sin\left(\frac{7\pi}{2}\right) = -1$$

That takes care of multiples of $\frac{\pi}{2}$ or π. For the rest, instead of trying to memorize the whole circle, you can divide the unit circle into two cases:

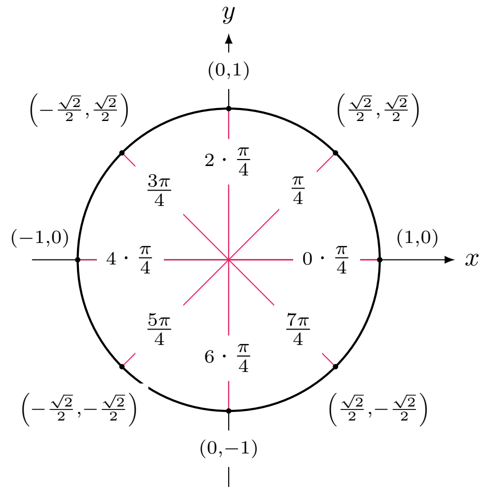
Multiples of $\frac{\pi}{4}$	Multiples of $\frac{\pi}{6}$ (and $\frac{\pi}{3}$)



Multiples of $\frac{\pi}{4}$:

Finding the positions: Each half of the circle has π radians, so if you divide the top and bottom halves of the unit circle into 4 wedges each, then every wedge covers $\frac{\pi}{4}$ radians. So to locate a multiple of $\frac{\pi}{4}$, divide each half into 4 wedges, and count counterclockwise.

Values of $\sin$ and $\cos$: At odd multiples of $\frac{\pi}{4}$, the values of $\sin$ and $\cos$ are always $\pm\frac{\sqrt{2}}{2}$. So you just have to determine whether the value of $\frac{\sqrt{2}}{2}$ is positive or negative. For this, just remember that $\cos$ is the x–value, $\sin$ is the y–value, and check whether that value is positive or negative in the quadrant you're in.

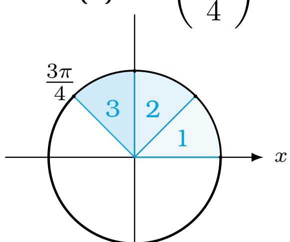


Exercise Find the following values, by first locating the angle on the unit circle:

(a) $\sin\left(\frac{3\pi}{4}\right)$ (b) $\cos\left(\frac{7\pi}{4}\right)$ (c) $\cos\left(\frac{9\pi}{4}\right)$ (d) $\sin\left(-\frac{\pi}{4}\right)$

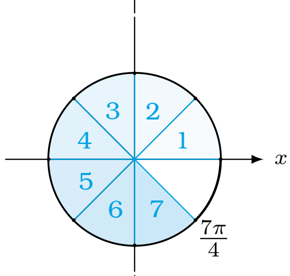
Solution

(a)



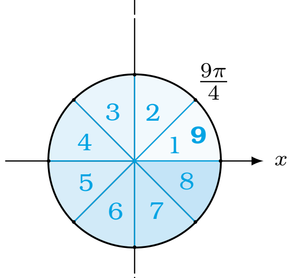
$\sin(\theta)$ is given by the y-value, which is positive in this position, so $\sin\left(\frac{3\pi}{4}\right) = \boldsymbol{\frac{\sqrt{2}}{2}}$.

(b)



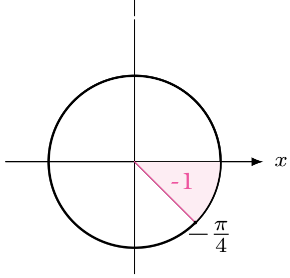
$\cos(\theta)$ is given by the x-value, which is positive in this position, so $\cos\left(\frac{7\pi}{4}\right) = \boldsymbol{\frac{\sqrt{2}}{2}}$.

(c)



$\cos(\theta)$ is given by the x-value, which is positive in this position, so $\cos\left(\frac{3\pi}{4}\right) = \boldsymbol{\frac{\sqrt{2}}{2}}$.

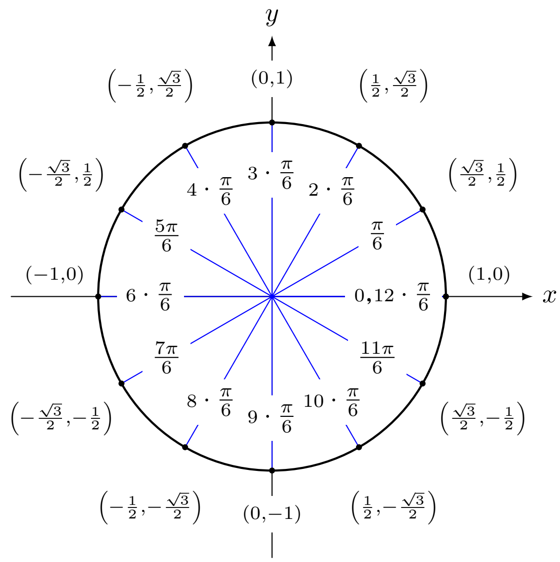
(d)



$\sin(\theta)$ is given by the y-value, which is negative in this position, so $\sin\left(-\frac{\pi}{4}\right) = \boldsymbol{-\frac{\sqrt{2}}{2}}$.

Multiples of $\frac{\pi}{6}$:

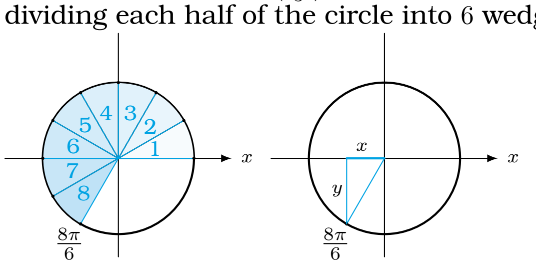
Finding the positions: First, keep in mind that any multiple of $\frac{\pi}{3}$ can be rewritten as a multiple of $\frac{\pi}{6}$, So for any question of the type $\frac{k\pi}{3}$, start by converting to a multiple of $\frac{\pi}{6}$.
Next, divide the top and bottom halves of the unit circle each in to 6 equal wedges. Each wedge is worth $\frac{pi}{6}$ radians. So to find a multiple of $\frac{\pi}{6}$, just count these wedges, moving counterclockwise for positive angles, and clockwise for negative angles.



Values of sin **and** cos**:** The coordinates at these points are always $\left(\pm\frac{1}{2},\pm\frac{\sqrt{3}}{2}\right)$, or $\left(\pm\frac{\sqrt{3}}{2},\pm\frac{1}{2}\right)$. So to determine the sin or cos value, you need to determine whether it's positive or negative, and whether it's $\frac{1}{2}$ or $\frac{\sqrt{3}}{2}$.

We can determine the sign of the $x-$ or $y-$ value from the quadrant of the graph we're in. Finally, just keep in mind that $\mathbf{\frac{1}{2}}$ **is smaller than** $\mathbf{\frac{\sqrt{3}}{2}}$ (since 1 is smaller than $\sqrt{3}$). So, locate the position of the angle, and look at whether the $x-$value is larger than y, or vice versa. The longer side is $\frac{\sqrt{3}}{2}$, the shorter one is $\frac{1}{2}$.

Example To find $\sin\left(\frac{4\pi}{3}\right)$, first I find the angle on the unit circle, by converting $\frac{4\pi}{3}$ to $\frac{8\pi}{6}$, dividing each half of the circle into 6 wedges, and counting 8 wedges counterclockwise:



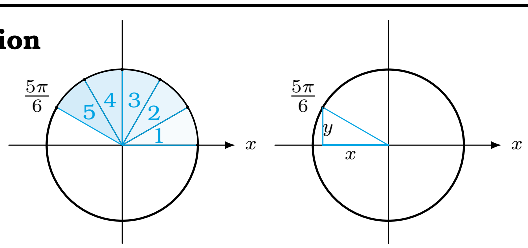
In this position, y is negative and y is larger than x, so the $y-$value has to be $-\frac{\sqrt{3}}{2}$. So $\sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2}$.

Exercise Find the following values, by first locating the angle on the unit circle:

(a) $\cos\left(\frac{5\pi}{6}\right)$ (b) $\cos\left(\frac{2\pi}{3}\right)$ (c) $\sin\left(-\frac{5\pi}{3}\right)$ (d) $\sin\left(\frac{7\pi}{3}\right)$

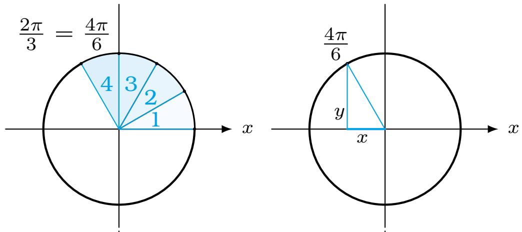
Solution

(a)



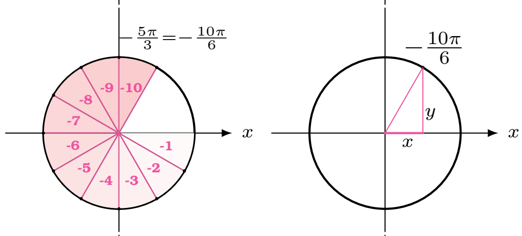
In this position, x is negative and x is larger than y, so the $x-$value has to be $-\frac{\sqrt{3}}{2}$. So $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$.

(b)



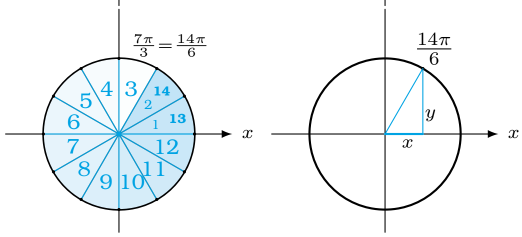
In this position, x is negative and x is smaller than y, so the x–value has to be $-\frac{1}{2}$. So $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$.

(c)



In this position, y is positive and y is larger than x, so the y–value has to be $\frac{\sqrt{3}}{2}$. So $\sin\left(-\frac{5\pi}{3}\right) = \frac{\sqrt{3}}{2}$.

(d)



In this position, y is positive and y is larger than x, so the y–value has to be $\frac{\sqrt{3}}{2}$. So $\sin\left(\frac{7\pi}{3}\right) = \frac{\sqrt{3}}{2}$.

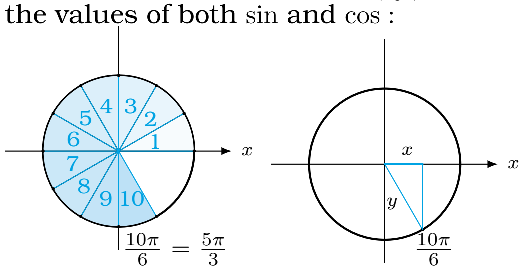
Other Trig Functions: To find values of $\tan(\theta)$, $\cot(\theta)$, $\sec(\theta)$ or $\csc(\theta)$, just convert them to expressions about $\sin$ and $\cos$, and follow the same method as above.

Recall:

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \qquad \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

$$\sec(\theta) = \frac{1}{\cos(\theta)} \qquad \csc(\theta) = \frac{1}{\sin(\theta)}$$

Example If I need to find $\tan\left(\frac{5\pi}{3}\right)$, I find the angle $\frac{5\pi}{3} = \frac{10\pi}{6}$ on the unit circle, and determine the values of both $\sin$ and $\cos$:



In this position, x is positive and y is negative, and y is larger than x. So $\sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}$, and $\cos\left(\frac{5\pi}{3}\right) = \frac{1}{2}$.

So then $\tan\left(\frac{5\pi}{3}\right) = \frac{\sin\left(\frac{5\pi}{3}\right)}{\cos\left(\frac{5\pi}{3}\right)} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}$