201-SH3-AB - Exercises #16: Probability

Numbers 1 to 5: Show that the function is a probability density function on the specified interval.

1. $f(x) = \frac{1}{16}x \qquad (2 \le x \le 6)$
2. $f(x) = \frac{3}{8}x^2 \qquad (0 \le x \le 2)$
3. $f(x) = 20(x^3 - x^4) \qquad (0 \le x \le 1)$
4. $f(x) = \frac{3}{14}\sqrt{x} \qquad (1 \le x \le 4)$
5. $f(x) = \frac{x}{(x^2+1)^{\frac{3}{2}}} \qquad (0 \le x < \infty)$

Numbers 6 to 9: Find the value of the constant k such that the function is a probability density function on the indicated interval.

6. $f(x) = k \qquad (1 \le x \le 4)$
7. $f(x) = k(4-x) \qquad (0 \le x \le 4)$
8. $f(x) = k\sqrt{x} \qquad (0 \le x \le 4)$
9. $f(x) = \frac{k}{x^3} \qquad (1 \le x < \infty)$

10. Given that $f(x) = k(4x - x^2)$ is a probability density function on the interval $(0 \le x \le 4)$,
    (a) Find the value of the constant k.
    (b) Suppose that X is a continuous random variable with probability density function f. Find the probability that X will assume a value between 1 and 3.

Numbers 11 to 14: Given that f is the probability density function for the random variable X defined on the given interval, find the indicated probabilities.

11. $f(x) = \frac{1}{12}x \qquad (1 \le x \le 5)$
    (a) $P(2 \le X \le 4)$ (b) $P(1 \le X \le 4)$ (c) $P(X \ge 2)$ (d) $P(X = 2)$

12. $f(x) = \frac{3}{32}(4 - x^2) \qquad (-2 \le x \le 2)$
    (a) $P(-1 \le X \le 1)$ (b) $P(X \le 0)$ (c) $P(X > -1)$ (d) $P(X = 0)$

13. $f(x) = \frac{1}{4\sqrt{x}} \qquad (1 \le x \le 9)$
    (a) $P(X \ge 4)$ (b) $P(1 \le X < 8)$ (c) $P(X = 3)$ (d) $P(X \le 4)$

14. $f(x) = 4xe^{-2x^2} \qquad (0 \le x < \infty)$
    (a) $P(0 \le X \le 4)$ (b) $P(X \ge 1)$

15. The average waiting time for patients arriving at one health clinic between 1 p.m. and 4 p.m. on a weekday is an exponential distributed random variable X with probability density function

$$f(x) = \frac{1}{15}e^{-x/100}; (0 \le x < \infty).$$

    (a) What is the probability that a patient arriving at the clinic between 1 p.m. and 4 p.m. will have to wait between 10 and 12 minutes?
    (b) What is the probability that a patient arriving at the clinic between 1 p.m. and 4 p.m. will have to wait more than 15 minutes?

16. The owner of a bakery finds that the waiting time (in minutes) for a customer to be served during the hours between 12 noon and 1 p.m. is an exponential distributed random variable X with associated probability density function

$$f(t) = \frac{1}{2}e^{-t/2}.$$

   (a) What is the probability that a customer arriving at the clinic between noon and 1 p.m. will have to wait at most 3 minutes?

   (b) What is the probability that a customer arriving at the clinic between noon and 1 p.m. will have to wait between 2 and 3 minutes?

   (c) What is the probability that a customer arriving at the clinic between noon and 1 p.m. will have to wait at least 3 minutes?

17. One welding company uses industrial robots in some of its assembly-line operations. Management has determined that the lengths of time (in hours) between breakdowns are exponentially distributed with probability density function

$$f(t) = 0.001e^{-0.001t}.$$

   (a) What is the probability that a robot selected at random will break down between 600 and 800 hours of use?

   (b) What is the probability that a robot selected at random will break down after 1200 hours of use?

18. A study conducted by a mail-order department store reveals that the time intervals (in minutes) between incoming telephone calls on its toll-free 800 line between 10 a.m. and 2 p.m. are exponentially distributed with probability density function $f(t) = \frac{1}{30}e^{-t/30}$.
What is the probability that the time interval between successive calls is more than 2 minutes?

19. The amount of rainfall (in inches) on a tropical island in the month of August is a continuous random variable with probability density function $f(x) = \frac{1}{16}x(6-x)$.
What is the probability that the amount of rainfall in August is less than 2 inches?

Numbers 20 to 25: Find the mean, variance, and the standard deviation of the random variable X associated with the probability density function over the indicated interval.

20. $f(x) = \frac{1}{3}$ $\quad(3 \le x \le 6)$

21. $f(x) = \frac{3}{125}x^2$ $\quad(0 \le x \le 5)$

22. $f(x) = \frac{3}{32}(x-1)(5-x)$ $\quad(1 \le x \le 5)$

23. $f(x) = \frac{8}{7x^2}$ $\quad(1 \le x \le 8)$

24. $f(x) = \frac{3}{14}\sqrt{x}$ $\quad(1 \le x \le 4)$

25. $f(x) = \frac{3}{x^4}$ $\quad(1 \le x < \infty)$

26. The amount of time (in minutes) a shopper spends browsing in the magazine section of a supermarket is a continuous random variable with probability density function $f(t) = \frac{2}{25}t$ where $(0 \le t \le 5)$.
How much time is a shopper chosen at random expected to spend in the magazine section?

27. The amount of time (in seconds) it takes a motorist to react to a road emergency is a continuous random variable with probability density function $f(t) = \frac{9}{4t^3}$ where $(1 \le t \le 3)$.
What is the expected reaction time for a motorist chosen at random?
Find the variance and the standard deviation.

28. The amount of snowfall (in feet) in a remote region in Alaska in the month of January is a continuous random variable with probability density function $f(t) = \frac{2}{9}x(3-x)$ where $(0 \le x \le 3)$.
Find the amount of snowfall one can expect in any given month of January in Alaska.
Find the variance and the standard deviation.

29. The lifespan (in years) of a certain brand of plasma TV is a continuous random variable with probability density function $f(t) = 9(9 + t^2)^{-3/2}$ where $(0 \leq t < \infty)$.
How long is one of these plasma TVs expected to last?

Numbers 30 to 32: Find the median of the random variable X associated with the probability density function over the indicated interval. Note that the median of X is defined to be the number m such that $P(X \leq m) = \frac{1}{2}$.

30. $f(x) = \frac{1}{6}$ $\qquad (2 \leq x \leq 8)$

31. $f(x) = \frac{3}{16}\sqrt{x}$ $\qquad (0 \leq x \leq 4)$

32. $f(x) = \frac{1}{x^2}$ $\qquad (1 \leq x < \infty)$

Numbers 33 to 38: Find the value of the probability of the standard normal random variable Z.

33. $P(Z < 1.45)$
34. $P(Z < -1.75)$
35. $P(-1.32 < Z < 1.74)$
36. $P(Z < -0.64)$
37. $P(Z > -1.26)$
38. $P(0.68 < Z < 2.02)$

39. Let Z be the standard normal variable. Find the values of z if z satisfies:

(a) $P(Z < z) = 0.8907$ (b) $P(Z < z) = 0.2090$ (c) $P(Z > -z) = 0.9713$ (d) $P(Z < -z) = 0.9713$

40. Let X be a normal random variable with $\mu = 80$ and $\sigma = 10$. Find the values of:

(a) $P(X < 100)$ (b) $P(X > 60)$ (c) $P(70 < X < 90)$

41. Let X be a normal random variable with $\mu = 50$ and $\sigma = 5$. Find the values of:

(a) $P(X < 60)$ (b) $P(X > 43)$ (c) $P(46 < X < 58)$

42. The serum cholesterol levels in milligrams per decaliter (mg/dL) in a current Mediterranean population are found to be normally distributed with a mean of 160 and a standard deviation of 50. Scientists at the National Heart, Lung, and Blood Institute consider this pattern ideal for a minimal risk of heart attacks. Find the percentage of the population having blood cholesterol levels between 160 and 180 mg/dL.

43. The medical records of infants delivered at one hospital show that the infants' length at birth (in inches) are normally distributed with a mean of 20 and a standard deviation of 2.6. Find the probability that an infant selected at random from among those delivered at the hospital measures:

(a) More than 22 inches. (b) Less than 18 inches. (c) Between 19 and 21 inches.

44. A certain company manufactures 50-,60-, 75-, and 100-watt light bulbs. Laboratory tests show that the lives of these light bulbs are normally distributed with a mean of 750 hours and a standard deviation of 75 hours. Find the probability that a light bulb selected at random from this company will burn:

(a) For more than 900 hours.
(b) For less than 600 hours.
(c) Between 750 and 900 hours.
(d) Between 600 and 800 hours.

45. The scores on a sociology examination are normally distributed with a mean of 70 and a standard deviation of 10. Suppose the instructor assigns letter grades to students in the class as follows:
Highest 15% get As ; next 25% get Bs ; next 40% get Cs; next 15% get Ds; lowest 5% get Fs
Find the cutoff points for grades A through D.

ANSWERS:

6. $k = \frac{1}{3}$
7. $k = \frac{1}{8}$
8. $k = \frac{3}{16}$
9. $k = 2$
10. (a) $k = \frac{3}{32}$
    (b) $\frac{11}{16}$
11. (a) $k = \frac{1}{2}$
    (b) $\frac{5}{8}$
    (c) $k = \frac{7}{8}$
    (d) 0
12. (a) $k = \frac{11}{16}$
    (b) $\frac{1}{2}$
    (c) $k = \frac{27}{32}$
    (d) 0
13. (a) $k = \frac{1}{2}$
    (b) $\frac{1}{2}(2\sqrt{2} - 1)$
    (c) 0
    (d) $k = \frac{1}{2}$
14. (a) 1
    (b) 0.135
15. (a) 0.06
    (b) 0.37
16. (a) 0.777
    (b) 0.145
    (c) 0.223
17. (a) 0.099
    (b) 0.30
18. 0.02
19. 0.5833
20. $\mu = \frac{9}{2}$; $Var(X) = \frac{3}{4}$; $\sigma \approx 0.8660$
21. $\mu = \frac{15}{4}$; $Var(X) = \frac{15}{16}$; $\sigma \approx 0.9682$
22. $\mu = 3$; $Var(X) = 0.8$; $\sigma \approx 0.8944$
23. $\mu \approx 2.3765$; $Var(X) \approx 2.3522$; $\sigma \approx 1.5337$
24. $\mu = \frac{93}{35}$; $Var(X) \approx 0.7151$; $\sigma \approx 0.846$
25. $\mu = \frac{3}{2}$; $Var(X) = \frac{3}{4}$; $\sigma = \frac{1}{2}\sqrt{3}$
26. $3\frac{1}{3}$ minutes
27. 1.5 seconds; $\frac{9}{4}(ln3 - 1)$; $\frac{3}{2}\sqrt{(ln3 - 1)}$
28. 1.5 feet; $\frac{9}{20}$; 0.671
29. 3 years
30. $m = 5$
31. $m \approx 2.52$
32. $m = 2$
33. 0.9265
34. 0.0401
35. 0.8657
36. 0.2611
37. 0.8962
38. 0.2266
39. (a) 1.23
    (b) -0.81
    (c) 1.9
    (d) -1.9
40. (a) 0.9772
    (b) 0.9772
    (c) 0.6826
41. (a) 0.9772
    (b) 0.9192
    (c) 0.7333
42. 0.1554
43. (a) 0.2206
    (b) 0.2206
    (c) 0.2960
44. (a) 0.0228
    (b) 0.0228
    (c) 0.4772
    (d) 0.7258
45. A: 80; B:73; C:62; D:54